Module 4A6 Flow induced sound and vibration

2003/04 Solutions

1 (i) The unsteady potential is $\phi = \frac{A}{r} e^{i\omega t - \frac{i\omega r}{c_0}} \equiv \frac{AE}{r}$.

1. Velocity $\underline{v} = \nabla\phi = \left(\frac{-i\omega}{c_0 r} - \frac{1}{r^2}\right) AE\, \hat{\underline{r}}$, ie purely radial.

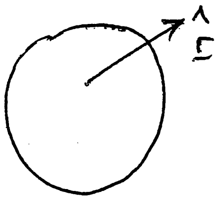
1. pressure $p = -\rho_0 \frac{\partial \phi}{\partial t} = -\frac{i\omega \rho_0 AE}{r}$, ρ_0 = mean density

Time-averaged energy flux vector

$$= \tfrac{1}{2} \mathrm{Re}\,(p^* v) = \tfrac{1}{2} \mathrm{Re}\left[\frac{i\omega \rho_0 A^* E^*}{r}\left\{\frac{-i\omega}{c_0 r} - \frac{1}{r^2}\right\} AE\right]\hat{\underline{r}}$$

In the farfield ignore $\frac{1}{r^2}$ term [since $r \gg c_0/\omega$, many wavelengths from source

1. $$\to \tfrac{1}{2} \mathrm{Re}\left[\frac{\omega^2 \rho_0 |A|^2}{c_0 r^2}\right]\hat{\underline{r}} = \frac{\omega^2 \rho_0 |A|^2}{2 c_0 r^2}\hat{\underline{r}}$$



Integrate over sphere of radius r. Normal is $\parallel$ to $\hat{\underline{r}}$, and surface area $= 4\pi r^2$

1. Thus, total power $= \frac{\omega^2 \rho_0 |A|^2}{2 c_0 r^2} \cdot 4\pi r^2 = \underline{\underline{\frac{2\pi \omega^2 \rho_0 |A|^2}{c_0}}}$.

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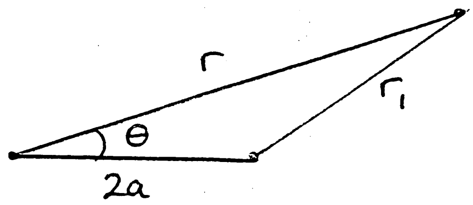
(ii)



Total potential $= \frac{A}{r} e^{i\omega t - \frac{i\omega r}{c_0}} + \frac{A}{r_1} e^{i4} e^{i\omega t - \frac{i\omega r_1}{c_0}}$ (*)

By the cosine rule, $r_1 = r\sqrt{1 - \frac{4a}{r}\cos\theta + \frac{4a^2}{r^2}}$

2. far from the source $\frac{a}{r} \ll 1$ $\Rightarrow r_1 \simeq r - 2a\cos\theta$.

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Qu1 cont.) In the second term in (*), approximate r_1 by r in the amplitude and by $r - 2a\cos\theta$ in the phase

$$(*) \rightarrow \frac{AE}{r}\left\{1 + \exp\left(\frac{2i\omega a\cos\theta}{c_0} + i\psi\right)\right\}$$

$$= \frac{2AE}{r} e^{i\omega a\cos\theta/c_0} e^{i\psi/2} \cos\left[\frac{\omega a\cos\theta}{c_0} + \frac{\psi}{2}\right].$$

Hence, pressure $p = -\rho_0 \frac{\partial\phi}{\partial t} = -\rho_0 i\omega\phi$

2. and so $|p| = \frac{2A\rho_0\omega}{r}\left|\cos\left(\frac{\omega a}{c_0}\cos\theta + \frac{\psi}{2}\right)\right|$

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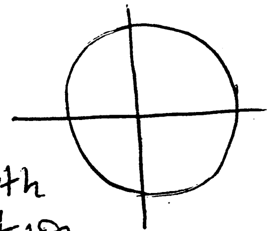
(iii) $\frac{\omega a}{c_0} \ll 1$ $\quad |p| \simeq \frac{2A\rho_0\omega}{r}|\cos\psi/2|$

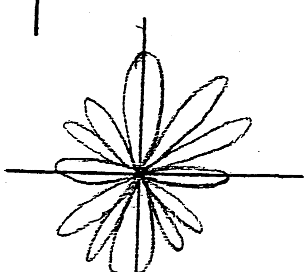


; independent of θ. Isotropic

2. sound field. In this case, wavelength is much longer than source separation, so far-field appears like that due to single source at origin

$\frac{\omega a}{c_0} \gg 1$ $\quad |p| \simeq \frac{2A\rho_0\omega}{r}\left|\cos\left(\frac{\omega a}{c_0}\cos\theta\right)\right|$

Many-lobed pattern, with



$O(\omega a/c_0)$ lobes. In this case

2. wavelength is much shorter than source separation, sources constructively interfere in some directions (maxima), destructively interfere in others (nulls).

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(iv) In the farfield, velocity approx. radial and $= p/\rho_0 c_0$ (plane-wave impedance).

1. Hence, total power $= \frac{1}{2}\mathrm{Re}\int_A p^* \frac{p}{\rho_0 c_0}\, dS$

Qu 1. cont.)

$$= \frac{1}{2\rho_0 c_0} 4|A|^2 \rho_0^2 \omega^2 \int_0^{\pi} \int_0^{2\pi} \cos^2\left(\frac{\omega a}{c_0}\cos\theta + \frac{\psi}{2}\right) \sin\theta \, d\phi \, d\theta$$

$$= \frac{2|A|^2 \rho_0 \omega^2}{c_0} \cdot 2\pi \int_0^{\pi} \sin\theta \left[\frac{1}{2}\cos\left(\frac{2\omega a}{c_0}\cos\theta + \psi\right) + \frac{1}{2}\right] d\theta$$

3. $$= \frac{4\pi|A|^2 \rho_0 \omega^2}{c_0} \left[-\frac{c_0}{4\omega a}\sin\left(\frac{2\omega a \cos\theta}{c_0} + \psi\right) - \frac{\cos\theta}{2}\right]_0^{\pi}$$

$$= \frac{4\pi|A|^2 \rho_0 \omega^2}{c_0} \left\{1 - \frac{c_0}{4\omega a}\left[\sin\left(-\frac{2\omega a}{c_0} + \psi\right) - \sin\left(\frac{2\omega a}{c_0} + \psi\right)\right]\right\}$$

$$= \frac{4\pi|A|^2 \rho_0 \omega^2}{c_0} \left\{1 + \frac{c_0}{2\omega a}\cos\psi \sin\left(\frac{2\omega a}{c_0}\right)\right\}$$

2. $$= \frac{4\pi|A|^2 \rho_0 \omega^2}{c_0} \left\{1 + \cos\psi \frac{\sin\left(\frac{2\omega a}{c_0}\right)}{\frac{2\omega a}{c_0}}\right\}$$

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When $\frac{\omega a}{c_0} \gg 1$, second term ignored. $\frac{\omega a}{c_0} \ll 1$, second term $\simeq 1$.

ratio of 2-source power to 1-source power $= 2\left[1 + \cos\psi \, \mathrm{sinc}\left(\frac{2\omega a}{c_0}\right)\right]$

When $2\omega a / c_0 \ll 1$ ratio $\simeq 2\left[1 + \cos\psi\right]$

1. Corresponds to 2 sources in same position. If $\psi = 0$, power increases by factor 4 (clear since pressure increases by factor 2). If $\psi = \pi$, total cancellation and no power

1. $2\omega a / c_0 \gg 1$ ratio $\simeq 2$

Two sources very far apart and uncorrelated, so powers additive.

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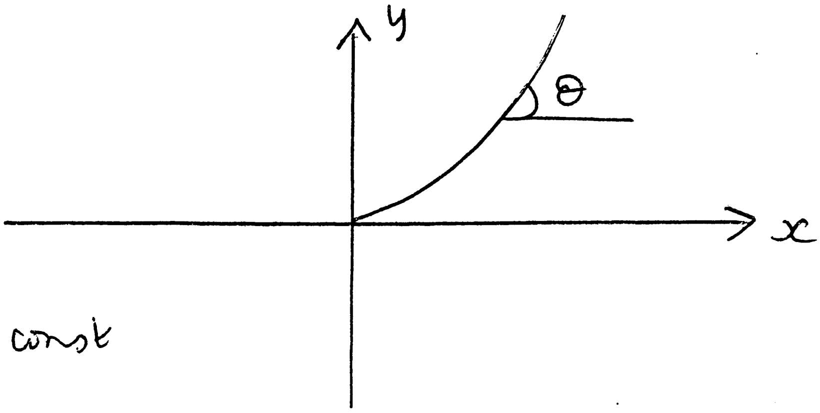
Qu 2(a)



(i)

$\frac{\sin\theta}{c} = \text{const}$

1. Launched from origin at angle $\theta = \theta_0$ $\quad \frac{\sin\theta}{c} = \frac{\sin\theta_0}{c_0}$

$$\sin\theta = \frac{dy/dx}{\sqrt{1+(dy/dx)^2}} \equiv y'/\sqrt{1+y'^2} \qquad \frac{y'^2}{c^2} = \frac{\sin^2\theta_0}{c_0^2}(1+y'^2)$$

$$\Rightarrow \quad y'^2 = \frac{c^2\sin^2\theta_0}{c_0^2} \Big/ 1 - \frac{c^2\sin^2\theta_0}{c_0^2}$$

2. $$y' = \frac{e^{\alpha x}\sin\theta_0}{\sqrt{1-e^{2\alpha x}\sin^2\theta_0}} \qquad \int dy = \sin\theta_0 \int \frac{e^{\alpha x}\,dx}{\sqrt{1-e^{2\alpha x}\sin^2\theta_0}}$$

Write $u = e^{\alpha x}\sin\theta_0$ $\quad \alpha\int dy = \int \frac{du}{\sqrt{1-u^2}}$

Now write $u = \sin\phi$ $\quad \alpha y = \phi + K \quad$ = constant

$$\alpha y = \sin^{-1}\{e^{\alpha x}\sin\theta_0\} + K$$

When $x = 0, y = 0$ $\quad \therefore K = -\sin^{-1}(\sin\theta_0) = -\theta_0$

3. $$y = \frac{1}{\alpha}\left\{\sin^{-1}(\sin\theta_0 e^{\alpha x}) - \theta_0\right\}$$

/6

(ii) The maximum value of the argument of $\sin^{-1}(z)$ is $z = 1$ $\quad \therefore$ $\sin\theta_0 e^{\alpha x}$ takes the maximum value of 1. $e^{\alpha x}$ increases with x, hence, maximum x is $e^{\alpha x} = \frac{1}{\sin\theta_0}$

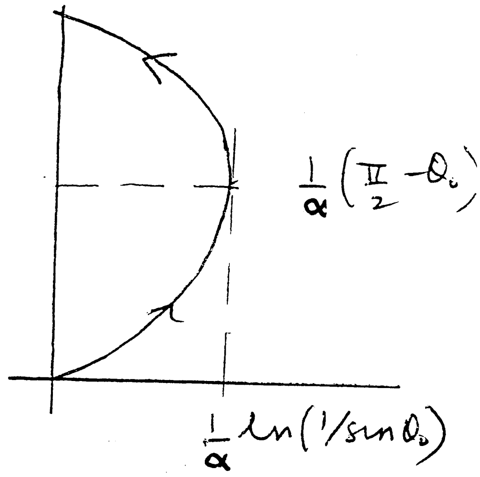
$$\Rightarrow \quad x = \frac{1}{\alpha}\ln\left[1/\sin\theta_0\right]$$

Qu2 cont.) for this value of x,

$$y = \frac{1}{\alpha}\left\{\sin^{-1}(1) - \theta_0\right\}$$

2.

$$= \frac{1}{\alpha}\left(\frac{\pi}{2} - \theta_0\right)$$



By symmetry of path, ray returns to $x=0$ at twice value

1. of maximum, i.e. at

$$y = \frac{2}{\alpha}\left(\frac{\pi}{2} - \theta_0\right)$$

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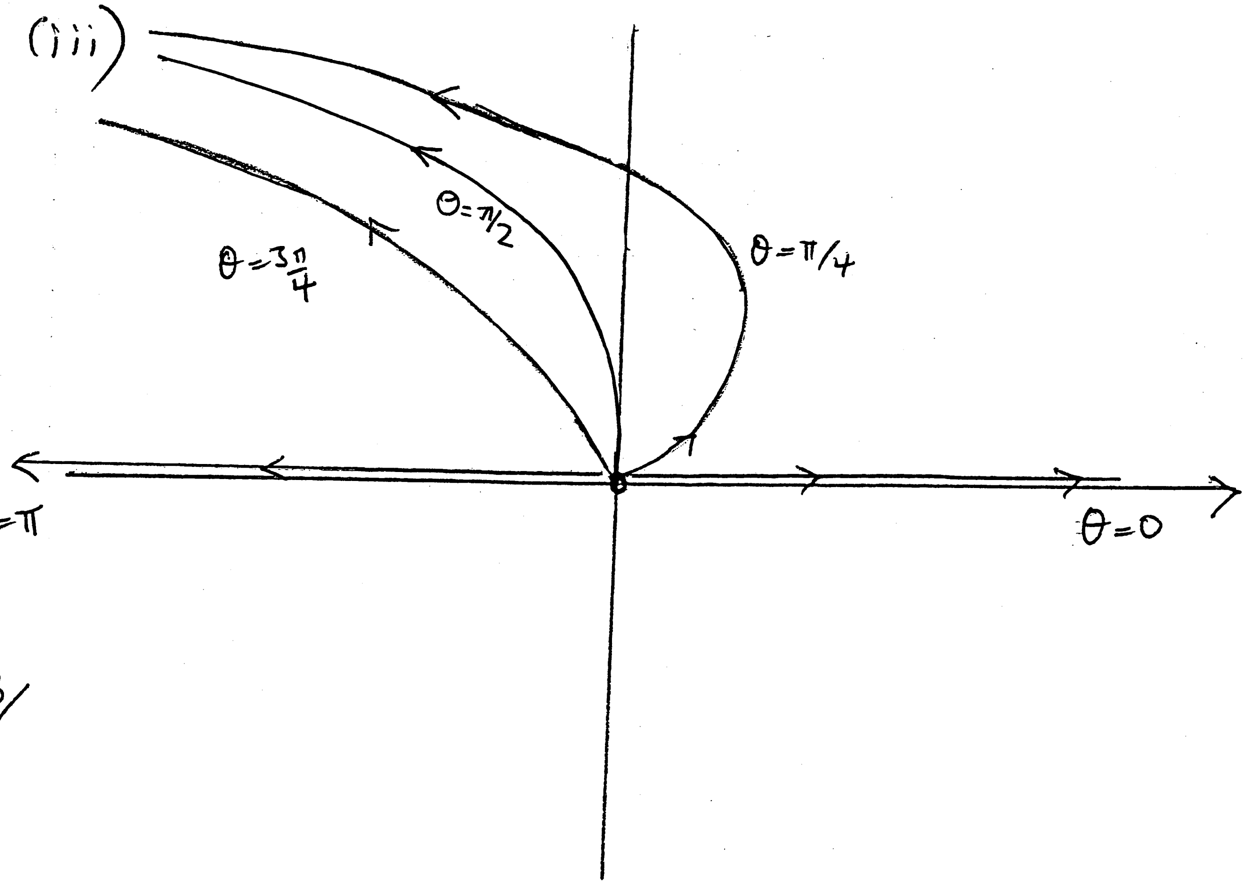
(iii)



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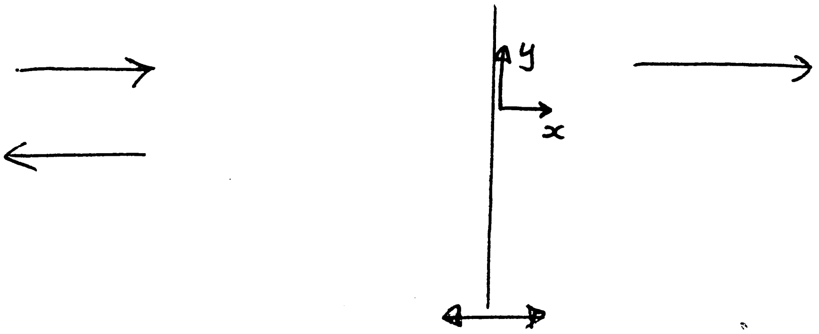
Qu2) cont. (b)



1. In $x<0$ $\quad p' = I e^{i\omega t - i\omega x/c_0} + R e^{i\omega t + i\omega x/c_0}$ (1)

1. In $x>0$ $\quad p' = T e^{i\omega t - i\omega x/c_0}$ (2)

Velocity of wall is $V e^{i\omega t}$

The eqn. of motion of the wall is $mVi\omega e^{i\omega t} = p(x=-0) - p(x=+0)$

1. $\Rightarrow \quad mi\omega V = -T + I + R$ (3)

Acoustic normal velocity on $x=-0$, $x=+0$ is, using

$$\frac{\partial p}{\partial x} = -\rho_0 \frac{\partial v}{\partial t} = -\rho_0 i\omega v \quad \Rightarrow \quad v = \frac{i}{\rho_0 \omega}\frac{\partial p}{\partial x}$$

$$\left.\begin{aligned} &\to \frac{(I-R)}{\rho_0 c_0} e^{i\omega t} \text{ for } x=-0 \\ &\to \frac{T e^{i\omega t}}{\rho_0 c_0} \text{ for } x=+0 \end{aligned}\right\} \quad (4)$$

2.

Further, these 2 speeds = speed of wall = $Ve^{i\omega t}$

$$\therefore \; I - R = T = \rho_0 c_0 V \quad (5)$$

Eliminate $R\ (=I-T)$ from (5) and $V\left(= T/\rho_0 c_0\right)$ from (5) in (3)

$$\frac{mi\omega T}{\rho_0 c_0} = -T + I + I - T$$

$$\Rightarrow \quad T = \frac{2I}{2 + \frac{im\omega}{\rho_0 c_0}} = \frac{2I\rho_0 c_0}{2\rho_0 c_0 + im\omega}.$$

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Qu 2 cont.)

2. Hence,

$$|T|/|I| = \frac{2\rho_0 c_0}{\sqrt{4\rho_0^2 c_0^2 + m^2\omega^2}}$$

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$$|T|/|I| = \frac{1}{\sqrt{1+\left(\frac{m\omega}{2\rho_0 c_0}\right)^2}}$$

$\frac{m\omega}{2\rho_0 c_0}$: wavelength $\lambda = \frac{2\pi}{k} = \frac{2\pi c_0}{\omega}$.

$$\therefore \frac{m\omega}{2\rho_0 c_0} = \frac{m\pi}{\rho_0\lambda}$$ $\rho_0\lambda$ = mass of air in one wavelength

$\therefore$ ratio is $\frac{\text{mass per unit area of wall}}{\text{mass of air in one wavelength}}$

1/

(20)

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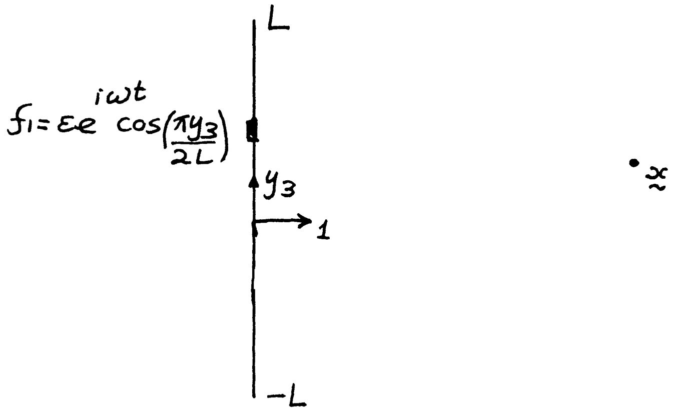
Qu 3.



a) The sound source is a dipole of strength f_1 /unit length

$$p'(\underset{\sim}{x},t) = -\frac{\partial}{\partial x_1}\int_{-L}^{L} \frac{f_1(y_3,\, t-|\underset{\sim}{x}-\underset{\sim}{y}|/c)}{4\pi|\underset{\sim}{x}-\underset{\sim}{y}|}\, dy_3$$

For $\underset{\sim}{x}$ in the far-field

$$p'(\underset{\sim}{x},t) = \frac{x_1}{4\pi|\underset{\sim}{x}|^2 c}\frac{\partial}{\partial t}\int_{-L}^{L} \varepsilon\cos\left(\frac{\pi y_3}{2L}\right) e^{i\omega(t-|\underset{\sim}{x}|/c + \underset{\sim}{x}\cdot\underset{\sim}{y}/|\underset{\sim}{x}|c)}\, dy_3$$

$$= \frac{i\omega x_1 \varepsilon}{4\pi|\underset{\sim}{x}|^2 c} e^{i\omega(t-|\underset{\sim}{x}|/c)}\int_{-L}^{L}\cos\left(\frac{\pi y_3}{2L}\right) e^{i\omega x_3 y_3/|\underset{\sim}{x}|c}\, dy_3$$

The integral is in the form of the hint with $\alpha = \frac{\omega x_3}{c|\underset{\sim}{x}|}$, $\beta = \frac{\pi}{2L}$

giving

$$p'(\underset{\sim}{x},t) = \frac{i\omega x_1\varepsilon}{4\pi|\underset{\sim}{x}|^2 c} e^{i\omega(t-|\underset{\sim}{x}|/c)} \frac{2}{\alpha^2-\beta^2}\left\{\alpha\sin(\alpha L)\cos(\beta L) - \beta\sin(\beta L)\cos(\alpha L)\right\}$$

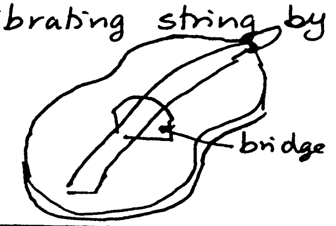
$$= -\frac{i\omega x_1\varepsilon}{4\pi|\underset{\sim}{x}|^2 c} e^{i\omega(t-|\underset{\sim}{x}|/c)} \frac{2}{\alpha^2-\pi^2/4L^2}\,\frac{\pi}{2L}\cos(\alpha L)$$

since $\cos\beta L = \cos\pi/2 = 0$
$\sin\beta L = 1$

$$= -\frac{i\omega x_1\varepsilon L}{|\underset{\sim}{x}|^2 c} e^{i\omega(t-|\underset{\sim}{x}|/c)} \frac{\cos(\alpha L)}{4\alpha^2 L^2 - \pi^2}$$

[80%]

b) The sound generated by this dipole is weak as indicated by the multiplying factor $\omega L/c$. To convert it into more efficient monopole sound could (i) use a sounding board coupled to the vibrating string by a bridge so that vibrations of the string are transferred to vibrations of a large surface eg violin, and/or (ii) place the string near an opening in a resonant cavity to excite fluctuating air flows in the opening. [20%]



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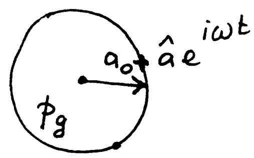
Qu 4.

a) (i) A monopole sound field is generated by a normal surface velocity over a sphere that is independent of position. [5%]

(ii) A dipole sound field is generated by a normal surface velocity proportional to $\cos\theta$ i.e. rigid body motion of the sphere. [5%]

b) The scattered field is monopole



$$p_s'(r,t) = \frac{A}{r} e^{+i\omega t - ikr} \qquad k = \omega/c$$

The change in radius of the bubble can be related to p_s and A through the momentum equation, $p_e = \bar{p}_0 + p_i + p_s$

$$-\rho_0 \omega^2 \hat{a} e^{i\omega t} = -\left.\frac{\partial p_s}{\partial r}\right|_{a_0} = A e^{i\omega t}\left(\frac{1}{a_0^2} + \frac{ik}{a_0}\right) e^{-ika_0}$$

$$\hat{a} = -\frac{A}{\rho_0 \omega^2 a_0^2}(1+ika_0) e^{-ika_0}$$

The air in the bubble responds adiabatically

$$p_g a^{3\gamma} = \text{constant}$$

$$p_g' a_0^{3\gamma} + p_{g0} a' 3\gamma a_0^{3\gamma-1} = 0$$

$$p_g' = -a' p_{g0} \frac{3\gamma}{a_0}$$

Surface tension

$$p_g - p_e = \frac{2T}{a}$$

$$\Rightarrow \quad p_{g0} = p_0 + \frac{2T}{a_0} \quad \text{and} \quad p_g' = p_e' - \frac{2T}{a_0^2} a'$$

$$-a'\left(p_0 + \frac{2T}{a_0}\right)\frac{3\gamma}{a_0} = p_i' + p_s' - \frac{2Ta'}{a_0^2}$$

Writing $p_i' = \hat{p}_i e^{i\omega t}$

$$0 = \hat{p}_i + \frac{A e^{-ika_0}}{a_0} + \left(p_0 \frac{3\gamma}{a_0} + \frac{2T(3\gamma-1)}{a_0^2}\right) a'$$

$$-\hat{p}_i e^{ika_0} a_0 = A\left(1 - \frac{(1+ika_0)}{\omega^2 a_0^2}\left(\frac{p_0}{\rho_0} 3\gamma + \frac{2T}{\rho_0 a_0}(3\gamma-1)\right)\right)$$

$$= A\left(1 - \frac{(1+ika_0)\omega_0^2}{\omega^2}\right) \quad \text{where} \quad \omega_0 = \left(\frac{p_0 3\gamma}{\rho_0 a_0^2} + \frac{2T}{\rho_0 a_0^3}(3\gamma-1)\right)^{1/2}$$

Hence

$$p_s'(r,t) = -\frac{\hat{p}_i e^{i\omega t - ik(r-a_0)}}{r\left(1 - (1+ika_0)\omega_0^2/\omega^2\right)}$$

[70%]

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Qu 4 cont.)

Surface tension raises the resonant frequency. [5%]
On resonance ($\omega = \omega_0$) the ika_0 in the denominator controls the level of the response and is due to the radiation of acoustic energy to the far-field. [5%]

(ii) If the scatterer is a rigid body, then it scatters sound due to the unsteady force it exerts on the fluid. The scattered field will therefore have a directional dependence and have a relatively weaker far-field. [10%]