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1. (a) DNS: very accurate, resolves all scales, very computer-intensive, still not possible at high Re

LES: Less costly than DNS, uses models for fine scales rather than resolving them, relies on accuracy of the models of small scales and on universality of these

RANS: Uses averaged equations, needs closure hypothesis for Reynolds stresses & relies heavily on the turbulence model used. Fast (relatively) and simple to use and included in commercial CFD packages. A number of parameters can be fixed by comparison with experiment.

(b) The Richardson number is the ratio of the thermal (buoyancy) generation of turbulence to the generation by mechanical shear. It is usually derived using eddy viscosity estimates.

If $Ri < 0$: unstable, turbulence increases

$Ri = 0$: neutral, no effect

$Ri > 0$: stable stratification, turbulence suppressed

($Ri > 1/4$: no turbulence)

(c) Dry Adiabatic Lapse Rate is the temperature variation with height that results in a stable situation. It is derived by considering adiabatic motion of a small parcel of fluid in an atmosphere whose pressure varies with height:

$\frac{dP}{dz} = -\rho g$ Using $T ds = dh - \frac{dp}{\rho}$, $dh = C_p dT$, $ds = 0$ (reversible adiabatic)

& $P = \rho R T$, gives $\frac{dT}{dz} = -\frac{g}{C_p}$ as the DALR.

(d) Katabatic flows are more intense since the stratification is stable (cold ground at night cools air above it): Mixing is suppressed and hence spreading is reduced. In anabatic flows turbulence is enhanced $\Rightarrow$ more spreading, which leads to a reduction in velocity

Both flows are driven by buoyancy (positive for anabatic, negative for katabatic).

2 (a) Density of parcel is constant ⇒ volume is constant

Buoyancy force $= (\rho_{ext} - \rho)\, g \cdot (\text{Volume})$

$$\rho_{ext} = \rho + \frac{\partial \rho}{\partial z} \Delta z$$

$$\Rightarrow \text{Force} = \frac{\partial \rho}{\partial z} \Delta z \,(\text{Vol}) \cdot g$$

$$= \frac{1}{\rho} \frac{\partial \rho}{\partial z} \Delta z \underbrace{\rho \,\text{Vol}}_{\text{mass}} g$$

$$\Rightarrow \text{force} = \frac{1}{\rho} \frac{\partial \rho}{\partial z} \Delta z \; mg$$

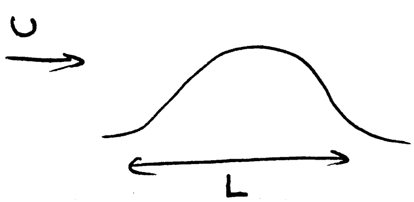
(b) In a spring-mass system $F = k \Delta z$

so $k = \frac{1}{\rho} \frac{\partial \rho}{\partial z} mg$

$$\Rightarrow \text{natural frequency} = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{\rho} \frac{\partial \rho}{\partial z} g}$$

(c) Internal waves usually arise when stratified fluid is perturbed by flowing over an obstacle. The perturbation starts the flow oscillating and these waves are transmitted through the fluid, since the displacement at one point leads to displacement of nearby fluid. The waves can carry energy to great heights. Waves are generally driven by changing in topography, e.g. hills.

Large internal waves occur when the driving frequency of the topography is equal to the natural frequency of the (stratified) flow (as derived in part b). The driving frequency depends on flow speed and size of hill, e.g.



$$f \propto \frac{U}{L}$$

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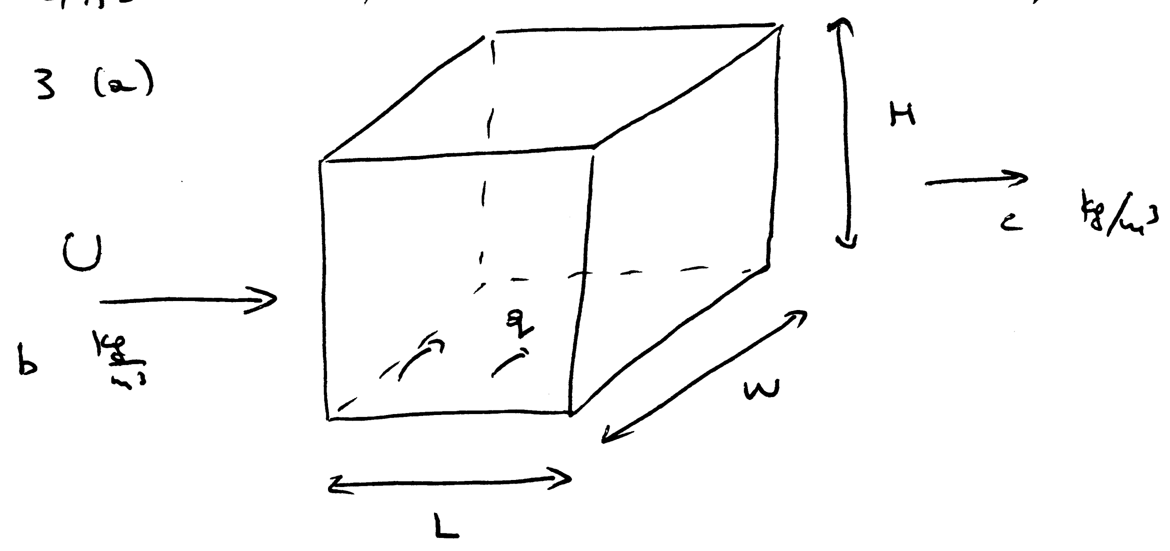
3 (a)



Conservation of mass of pollutant:

$$\underbrace{\frac{d(c.LWH)}{dt}}_{\text{accumulation}} = \underbrace{UbWH}_{\text{kg of pollutant coming in per second}} - \underbrace{UcWH}_{\text{kg of pollutant leaving per s}} + \underbrace{qLW}_{\text{emitted kg/s}} + \underbrace{\dot{w}LWH}_{\text{reaction}}$$

For $b=0$ (uncontaminated upwind air), $\dot{w}=0$ (inert) & steady-state $\left(\frac{dc}{dt}=0\right)$ means:

$$UcWH = qLW \Rightarrow c = \frac{qL}{UH}$$

(b) c at night would be high because H (mixing height) would be low. At sunrise, stability close to ground changes & a well-mixed region develops & grows (i.e. H ↗). Hence, c ↘ as time progresses.

Throughout the day, H would stay high $\Rightarrow c$ low.

In practice, emissions are higher during the day (traffic, industrial activity etc) $\Rightarrow$ c does not necessarily follow the above argument.

(c) If another S kg s^{-1} m^{-3} are emitted, then the governing equation includes a term $S \cdot HWL$ in the right-hand-side. Hence

$$\frac{dc}{dt} = \frac{Ub}{L} - \frac{Uc}{L} + \frac{q}{H} + S.$$

At steady state & $b=0$, $c = \dfrac{qL}{UH} + \dfrac{SL}{U}$

If S is a random variable with mean $\bar{S}$ & variance σ^2, we can think of c as a function of S

$$\Rightarrow \quad \bar{c} = \int f(S)\,P(S)\,dS, \qquad f(S) = \frac{qL}{UH} + S\frac{L}{U}$$

$P(S) =$ PDF of S

$$= \frac{qL}{UH}\int P(s)ds + \frac{L}{U}\int s\,P(s)ds$$

$$\Leftrightarrow \quad \boxed{\bar{c} = \frac{qL}{UH} + \frac{L}{U}\bar{S}}$$

For the variance:

$$\sigma_c^2 = \int (c-\bar{c})^2 P(S)dS = \int [f(s)-\bar{c}]^2 P(s)ds$$

$$= \int \left(\frac{qL}{UH} + \frac{SL}{U} - \frac{qL}{UH} - \frac{L\bar{S}}{U}\right)^2 P(s)ds$$

$$= \frac{L^2}{U^2}\int (S-\bar{S})^2 P(s)ds$$

$$\Leftrightarrow \quad \boxed{\sigma_c^2 = \frac{L^2}{U^2}\sigma^2}$$

The PDF of c will also be gaussian because f(s) is linear. To show this mathematically the starting point is that

P that s lies between S & S+ds = P(s)ds

= P that c lies between c & c+dc = P(c)dc

$$\Rightarrow \quad P(s)\,ds = P(c)\,dc \iff P(c) = P(s)\frac{ds}{dc}$$

Since $c = \frac{qL}{UH} + \frac{LS}{U}$, $S = \left(c - \frac{qL}{UH}\right)\frac{U}{L}$

$$\Rightarrow \quad \frac{ds}{dc} = \frac{U}{L}$$

If $P(s) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(s-\bar{s})^2}{2\sigma^2}\right)$,

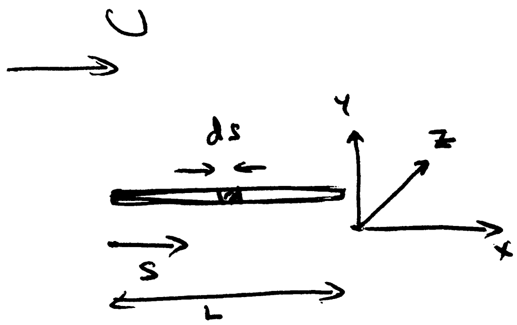
$$P(c) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{\left(c\frac{U}{L} - \frac{qL}{UH} - \frac{\bar{c}U}{L} + \frac{qL}{UH}\right)^2}{2\sigma^2}\right]\frac{U}{L}$$

$$= \frac{1}{\sqrt{2\pi}\;\frac{\sigma L}{U}} \exp\left[-\frac{(c-\bar{c})^2}{2\left(\frac{\sigma^2 L^2}{U^2}\right)}\right]$$

$$\Rightarrow \quad P(c) = \frac{1}{\sqrt{2\pi}\,\sigma_c} \exp\left[-\frac{(c-\bar{c})^2}{2\sigma_c^2}\right] \qquad \left(\sigma_c = \sigma\frac{L}{U} \text{ from before}\right)$$

ie. a gaussian.

4 (a)



s = distance along line source

Consider an infinitesimal point source at s. Its strength is $Q \cdot ds$

$\Rightarrow$ due to <u>this</u> source

$$d\phi(x,y,z) = \frac{Q\,ds}{2\pi U \sigma_y \sigma_z} \exp\left[-\frac{y^2}{2\sigma_y^2}\right] \exp\left[-\frac{z^2}{2\sigma_z^2}\right]$$

$$\sigma_z = \sigma_y = C\,(\underbrace{x+L-s}_{=\text{distance from source}})$$

$\Rightarrow$ due to all sources: (linear superposition allowed because pollutant is inert)

$$\phi(x,0,0) = \int_0^L \frac{Q\,ds}{U 2\pi\, C^2 (x+L-s)^2}$$

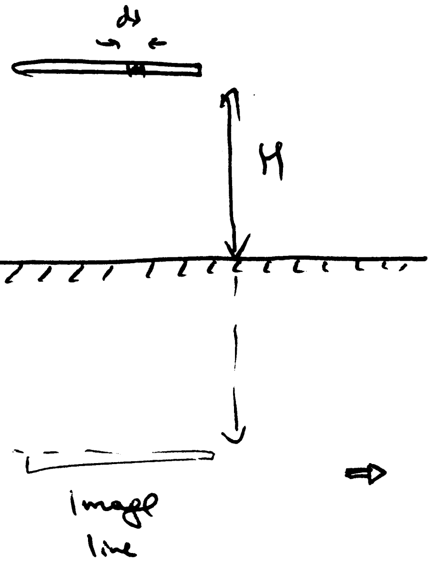
$$\Rightarrow \phi(x,0,0) = \frac{Q}{2\pi U C^2} \int_0^L \frac{ds}{(x+L-s)^2} = \frac{Q}{U 2\pi C^2}\left[\frac{1}{x+L-s}\right]_0^L$$

$$= \frac{Q}{2\pi C^2 U}\left(\frac{1}{x} - \frac{1}{x+L}\right)$$

$$= \frac{Q}{U 2\pi C^2}\,\frac{x+L-x}{x(x+L)} = \frac{QL}{U 2\pi\, C^2 x(x+L)}$$

(Check: QL : kg/s emitted totally.
As $x \gg L$ $\quad x(x+L) \approx x^2 \Rightarrow$ result is $\frac{(QL)}{U 2\pi C^2 x^2}$
as it should from a single source)

(b) If line source is H above ground:



ϕ due to infinitesimal source

$$d\phi(x,y,z) = \frac{Q\,ds}{U\,2\pi\sigma_y\sigma_z}\left[e^{-y^2/2\sigma_y^2}\right]\left[e^{-z^2/2\sigma_z^2} + e^{-\frac{(z+2H)^2}{2\sigma_z^2}}\right]$$

$$\Rightarrow d\phi(x,0,0) = \frac{Q\,ds}{U\,2\pi\sigma_y\sigma_z}\left(1 + e^{-\frac{4H^2}{2\sigma_z^2}}\right)$$

Now, upon integration $\phi(x,0,0) = \displaystyle\int_0^L \frac{Q\,ds}{U\,2\pi\,C^2(x+L-s)^2}\left(1+e^{-\frac{4H^2}{2C^2(x+L-s)^2}}\right)$

this integral cannot be evaluated analytically.

Important point is that σ_z appears inside integral in $e^{-4H^2/2\sigma_z^2}$ term $\Rightarrow$ difficult to integrate.