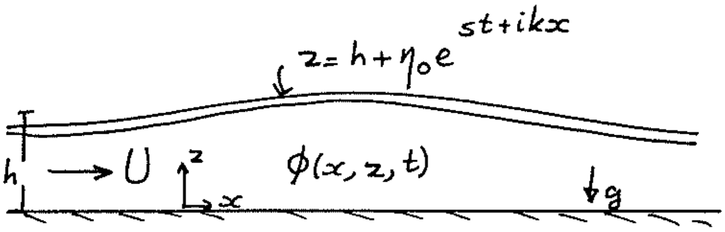
Module 4A10 Flow Instability 2004

Qu 1.



(i) The velocity potential has the form:

$$\phi(x,z,t) = Ux + f(z)e^{st+ikx}$$

$$\nabla^2\phi = 0 \Rightarrow \frac{d^2f}{dz^2} - fk^2 = 0$$

$$f(z) = C_1 e^{kz} + C_2 e^{-kz}$$

$\frac{d\phi}{dz} = 0$ on $z = 0$,

$$0 = C_1 - C_2$$

i.e. $f(z) = 2C_1\cosh(kz) = A\cosh(kz)$ say, for some complex constant A.

$$\underline{\underline{\phi(x,z,t) = Ux + A\cosh(kz)e^{st+ikx}}}$$

(ii) <u>Kinematic boundary condition</u>

On $z=h+\eta$, $\frac{\partial\phi}{\partial z} = \frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x}$ after linearising

$$Ak\sinh(kh) = (s+ikU)\eta_0 \qquad (1)$$

<u>Pressure on $z=h+\eta$,</u> is given by unsteady Bernoulli equation:

$$p + \rho\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\rho u^2 + \rho g z = \text{constant} = p_0 + \tfrac{1}{2}\rho U^2 + \rho g h$$

$$p\big|_{h+\eta} = -\rho\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho u^2 - \rho g\eta + p_0 + \tfrac{1}{2}\rho U^2$$

$$p\big|_{h+\eta} = \left(-\rho(s+ikU)A\cosh(kh) - \rho g\eta_0\right)e^{st+ikx}$$

<u>Equation of motion of sheet</u>

$$p - p_a = m\frac{\partial^2\eta}{\partial t^2} = ms^2\eta_0 e^{st+ikx}$$

i.e. $-\rho(s+ikU)A\cosh(kh) - \rho g\eta_0 = ms^2\eta_0$

$$-\rho(s+ikU)A\cosh(kh) = (\rho g + ms^2)\eta_0$$

Substituting for η_0 from equation (1) gives

$$-\rho(s+ikU)A\cosh(kh) = \frac{(\rho g + ms^2)Ak\sinh(kh)}{s+ikU}$$

Qu1 cont.)

$$-(s+ikU)^2 = k\tanh(kh)\left(\frac{ms^2}{\rho}+g\right)$$

$$0 = s^2\left(1+\frac{mk\tanh(kh)}{\rho}\right) + 2sikU + \left(-U^2k^2 + k\tanh(kh)g\right)$$

This is a quadratic equation for s, and has a root with a positive real part only if

$$(2ikU)^2 > 4\left(1+\frac{mk}{\rho}\tanh(kh)\right)\left(-U^2k^2 + k\tanh(kh)g\right)$$

i.e. unstable only if

$$+k^2U^2\frac{mk}{\rho}\tanh(kh) > \left(1+\frac{mk}{\rho}\tanh(kh)\right)k\tanh(kh)g$$

$$\underline{\underline{U^2 > \frac{g}{k}\left(\frac{\rho}{mk}+\tanh(kh)\right)}} \qquad (2)$$

(iii) Consider right-hand side of equation (2),

When $k=0$, $RHS=\infty$

$k\to\infty$, $RHS\to\infty$

it is always positive and so it has a minimum

Only have instability when $U^2 >$ this minimum.

To cast equation (2) in nondimensional form, write $X = hk$

$$U^2 > \frac{gh}{X}\left(\frac{\rho h}{m}+\tanh(X)\right) \quad \text{for instability at wavenumber } k=\frac{X}{h}$$

$$\frac{U_c^2}{gh} = \min_{0\le X\le\infty}\left(\frac{1}{X}\left(\frac{\rho h}{m}+\tanh(X)\right)\right)$$

To find U_c, we need to find the minimum of $\frac{1}{X}\left(\frac{\rho h}{m}+\tanh(X)\right)$

This is clearly a function of $(\rho h/m)$ ie $\underline{\underline{\frac{U_c^2}{gh} = F\left(\frac{\rho h}{m}\right)}}$.

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Qu 2.

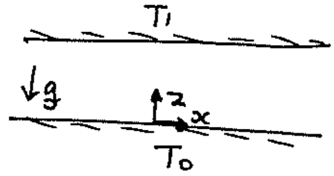
The Boussinesq approximation

neglects any variation of ν, κ and α with temperature,

neglects any variation of density with temperature in the continuity or inertia term ie. $\rho \frac{D\underset{\sim}{u}}{Dt}$ becomes $\rho_0 \frac{D\underset{\sim}{u}}{Dt}$,

but retains the variation of density in the buoyancy term $-\rho g$, where it is linearised, ie $\rho = \rho_0(1-\alpha(T-T_0))$

(i). In the equilibrium configuration



$\underset{\sim}{u} = \underset{\sim}{0}$, $\quad \nabla^2 \bar{T} = 0$

with solution $\bar{T} = T_0 + (T_1 - T_0) z/d$

$$\frac{d\bar{p}}{dz} = -\rho_0 (1 - \alpha(\bar{T} - T_0)) g$$

$$= -\rho_0 \left(1 - \alpha(T_1 - T_0) \frac{z}{d}\right) g$$

$$\bar{p}(z) = \bar{p}_0 - \rho_0 g \left(z - \alpha(T_1 - T_0)\frac{z^2}{2d}\right)$$ where $\bar{p}_0$ = mean pressure of $z=0$.

(ii) If a fluid particle is raised adiabatically, its temperature and hence its density are less than the surrounding particles. Hence the upward buoyancy force exceeds the weight of the particle and so it continues to move upward $\rightarrow$ **instability**.

(iii) For linearised perturbations from the equilibrium configuration the Boussinesq approximations to the equations of motion simplify to

$$\nabla \cdot \underset{\sim}{u} = 0$$

$$\frac{\partial \underset{\sim}{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p' - \alpha T' \underset{\sim}{g} + \nu \nabla^2 \underset{\sim}{u}$$

$$\frac{\partial T'}{\partial t} + w\frac{d\bar{T}}{dz} = \kappa \nabla^2 T' \Rightarrow \frac{\partial T'}{\partial t} + \frac{w}{d}(T_1 - T_0) = \kappa \nabla^2 T'$$

where p', T' denote perturbations from the equilibrium configuration, w = velocity in z-direction.

Nondimensional lengths on d, time on d^2/ν, temperature on $1/\alpha$ (note there are many ways of doing the nondimensionalisation) Then with ~ denoting the nondimensional variables:

$$\tilde{\nabla} \cdot \tilde{\underset{\sim}{u}} = 0$$

Non-dimensionalise velocity on ν/d, pressure on $\rho_0 \nu^2/d^2$:

Momentum eqn $\Rightarrow \frac{\nu}{d}\frac{\nu}{d^2}\frac{\partial \tilde{\underset{\sim}{u}}}{\partial \tilde{t}} = -\frac{1}{\rho_0}\rho_0\frac{\nu^2}{d^2}\frac{1}{d}\tilde{\nabla}\tilde{p}' - \frac{\alpha}{\alpha} g \tilde{T}' + \nu\frac{\nu}{d}\frac{1}{d^2}\tilde{\nabla}^2\tilde{\underset{\sim}{u}}$

Qu2 cont.)
$$\frac{\partial \tilde{\underline{u}}}{\partial \tilde{t}} = -\nabla \tilde{p}' + \frac{g d^3}{\nu^2} \underline{e}_z \tilde{T}' + \tilde{\nabla}^2 \tilde{\underline{u}} .$$

Heat flux eqn.
$$\frac{\nu}{d^2} \frac{1}{\alpha} \frac{\partial \tilde{T}'}{\partial \tilde{t}} + \frac{\nu}{d} \frac{1}{d} \tilde{w} (T_1 - T_0) = \frac{\kappa}{d^2 \alpha} \tilde{\nabla}^2 \tilde{T}'$$
$$\Rightarrow \quad \frac{\partial \tilde{T}'}{\partial \tilde{t}} + \alpha (T_1 - T_0) \tilde{w} = \frac{1}{Pr} \tilde{\nabla}^2 \tilde{T}' \qquad Pr = \nu/\kappa .$$

This is not yet in the form we want, which to have the equations depending only on $\underline{\underline{2}}$ nondimensional groups, Rayleigh number and Prandtl number, currently we have $\underline{\underline{3}}$. However, 2 can be combined by introducing a new temperature-like variable
$$\tilde{\Theta}' = \tilde{T}' \frac{g d^3}{\nu^2}$$

Then the momentum equation becomes
$$\frac{\partial \tilde{\underline{u}}}{\partial \tilde{t}} = -\nabla \tilde{p}' + \underline{e}_z \tilde{\Theta}' + \tilde{\nabla}^2 \tilde{\underline{u}},$$
and the heat flux equation becomes
$$\frac{\partial \tilde{\Theta}'}{\partial \tilde{t}} + \alpha . (T_1 - T_0) g \frac{d^3}{\nu^2} \tilde{w} = \frac{1}{Pr} \tilde{\nabla}^2 \tilde{T}'$$
$$Pr \frac{\partial \tilde{\Theta}'}{\partial \tilde{t}} + \alpha g d^3 (T_1 - T_0) \frac{1}{\nu^2} \frac{\nu}{\kappa} \tilde{w} = \nabla^2 \tilde{T}'$$
i.e. $$Pr \frac{\partial \tilde{\Theta}'}{\partial \tilde{t}} + Ra\, \tilde{w} = \nabla^2 \tilde{T}'$$

Note: different nondimensionalisations will lead to different forms of the equations, but the important thing is that they can be reduced just to involve Ra & Pr

(iv) The Rayleigh number is the ratio of the buoyancy driving mechanism proportional to $g\alpha(T_1 - T_0)$, opposed by the frictional action of viscosity and by the diffusion of the temperature difference described by ν and κ respectively.

(v) After the first critical flow condition a new steady flow occurs. This onset is therefore not influenced by the Prandtl number which only appears multiplying a time derivative.

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3.

a) $$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial V_1}\frac{dV_1}{dt} + \frac{\partial \phi}{\partial V_2}\frac{dV_2}{dt}$$

where $\phi = a_1 r\cos\theta + \frac{a_2}{r}\cos\theta$

and $a_1 = \frac{V_2 R_2^2 - V_1 R_1^2}{(R_2^2 - R_1^2)}$ and $a_2 = \frac{(V_2 - V_1)R_1^2 R_2^2}{(R_2^2 - R_1^2)}$

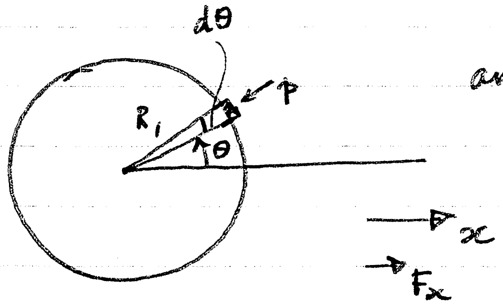
hence $\frac{\partial \phi}{\partial V_1} = \frac{-R_1^2}{R_2^2 - R_1^2} r\cos\theta - \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\frac{\cos\theta}{r}$

and $\frac{\partial \phi}{\partial V_2} = \frac{R_2^2}{R_2^2 - R_1^2} r\cos\theta + \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\frac{\cos\theta}{r}$

so $$\frac{\partial \phi}{\partial t} = \left\{\frac{-R_1^2}{R_2^2 - R_1^2} r\cos\theta - \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\frac{\cos\theta}{r}\right\}\frac{dV_1}{dt} \ldots$$

$$\ldots + \left\{\frac{R_2^2}{R_2^2 - R_1^2} r\cos\theta + \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\frac{\cos\theta}{r}\right\}\frac{dV_2}{dt}$$ 2 marks

b) consider the pressure around the central cylinder and ~~its~~ the horizontal force it causes:



area = $R_1 d\theta$

F_x on element $= -p\cos\theta R_1 d\theta$

total force on pipe $= \rho \int_{\theta=0}^{2\pi} \left(\frac{\partial \phi}{\partial t}\right)\cos\theta R_1 d\theta$

$\Rightarrow$ force on pipe $= \rho \displaystyle\int_0^{2\pi} \frac{\partial \phi}{\partial t} \cos\theta \, R_1 \, d\theta$ 3 for identifying how to do it

$$= \rho R_1^2 \int_0^{2\pi} \left(\frac{-R_1^2}{R_2^2 - R_1^2} \cos^2\theta - \frac{\cancel{R_1} R_2^2}{R_2^2 - R_1^2} \cos^2\theta \right) \frac{dV_1}{dt} \, d\theta \quad \ldots$$

$$\ldots + \rho R_1^2 \int_0^{2\pi} \left(\frac{R_2^2}{R_2^2 - R_1^2} \cos^2\theta + \frac{R_2^2}{R_2^2 - R_1^2} \cos^2\theta \right) \frac{dV_2}{dt} \, d\theta$$ 1 for substitution

since $\displaystyle\int_0^{2\pi} \cos^2\theta \, d\theta = \pi$, we can write 1 for evaluation

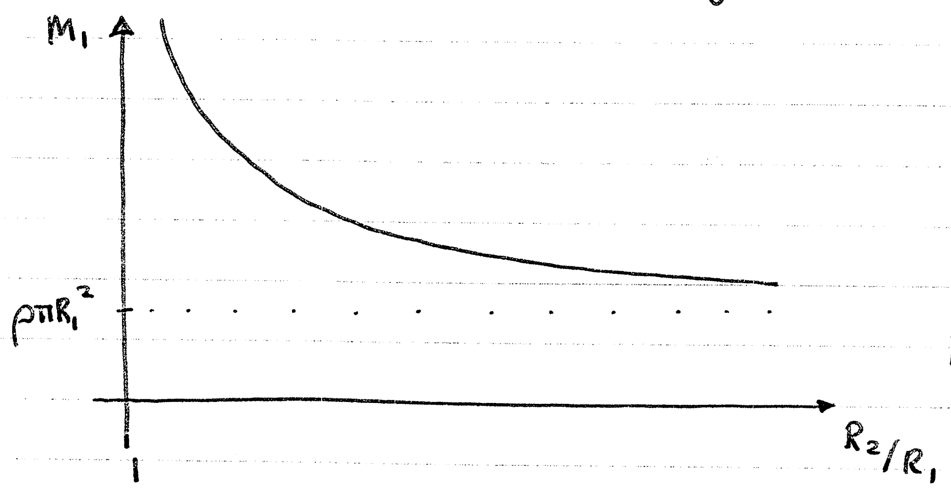
$$F_1 = -\rho\pi R_1^2 \left\{ \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \right\} \frac{dV_1}{dt} + \rho\pi R_2^2 \left\{ \frac{R_1^2}{R_2^2 - R_1^2} + \frac{R_1^2}{R_2^2 - R_1^2} \right\} \frac{dV_2}{dt}$$ 2 for working

hence $\displaystyle F_1 = -M_1 \frac{dV_1}{dt} + M_{12} \frac{dV_2}{dt}$

where $\displaystyle M_1 = \rho\pi R_1^2 \left\{ \frac{(R_2/R_1)^2 + 1}{(R_2/R_1)^2 - 1} \right\}$ and $\displaystyle M_{12} = \rho\pi R_2^2 \left\{ \frac{2}{(R_2/R_1)^2 - 1} \right\}$

1 for presenting in correct format.

c) M_1 represents the added mass of the central cylinder caused by interaction with the fluid between the two cylinders. 1 mark



1 mark.

d) When the central tube is full, it contains fluid of mass: $\rho\pi R_1^2 = m_1$
At this mass, the tube has a natural frequency $f_1 = 100\,Hz$.
Assuming a mass-spring system models this sufficiently:

$$f_1 \sim \sqrt{\frac{k}{m_1}}$$

~~3~~ 1 marks for method

where k is a spring constant.

2 marks for method

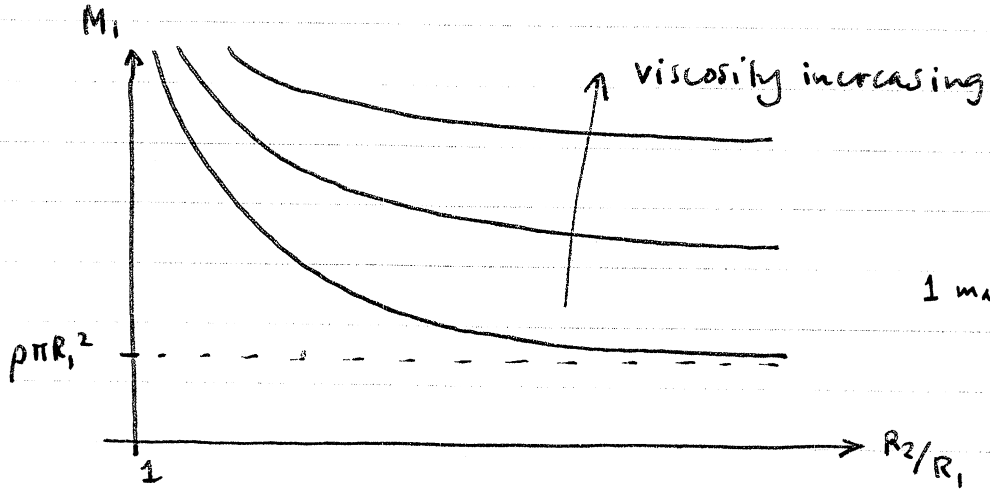
When the space between the tubes is full and $R_2/R_1 = 2$, the added mass is $M_1 = \rho\pi R_1 \left(\frac{5}{3}\right)$. Consequently, the ~~central~~ fluid forces on the central tube can be modelled ~~as an ad~~ by giving ~~tube can be modelled~~ the central tube a mass $m_2 = m_1 + M_1 = 8/3\, m_1$

Consequently $$\frac{f_2}{f_1} = \sqrt{\frac{m_1}{m_2}} = \sqrt{\frac{3}{8}}$$

1 mark for answer

So the natural frequency will be $100\,Hz * \sqrt{\frac{3}{8}} = 61\,Hz$

e)



1 mark for graph

- As the viscosity increases the natural frequency will decrease further.

1 mark for either.

- There will be an added damping coefficient too.

f) M_{12} represents the fluid coupling between the outer and the inner cylinder.

2 marks for this or equivalent answer.

Module 4A10 Flow Instability 2004 exam solutions

4.

a) Motion in towers & walkway

i) Wind from direction A - motion will be in the towers.

- Vortices will be shed from the sides of the upstream tower. These could be from alternate sides (vortex street) or from both sides simultaneously. (1 mark)

- The vortex shedding frequency can be estimated e.g. for alternate sides: $St = \frac{fD}{U} \simeq 0.3$ for these Reynolds numbers ($\sim 10^8$).

if $U = 20\,ms^{-1}$ and $D \simeq 50m$, $f = \frac{0.3 \times 20}{50} = 0.12$

ie. a cycle every 8 seconds.

The vortex shedding frequency for the other mode will be the same order of magnitude.

- The shedding frequency <u>may</u> lock into ~~the~~ a natural frequency of the tower. This will depend on the ovalling and swaying frequencies of the tower (which should be estimated or measured by the designers). If lock-in occurs, the amplitude of oscillation of the upstream tower will increase and vortices will be shed at its natural frequency (rather than the normal vortex shedding frequency). (1 mark)

- The shed vortices will impinge on the downstream tower and will force it at <u>exactly</u> its natural frequency. It will, consequently, vibrate even more than the upstream tower. (This is what happened at Ferrybridge Power station when they built the cooling towers too close together) (1 mark)

- To prevent this motion you could try to de-correlate the vortex shedding along the length of the towers with a helical strake or prevent shedding entirely with a streamlined fairing (this would not be practical). (1 mark)

+ 1 discretionary mark for excellent answer.

cont ...

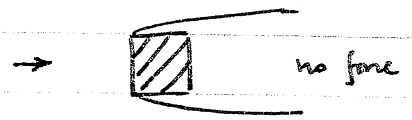
• The stiffness and mass of the towers could also be changed in order to change the natural frequencies of the towers. This would prevent lock-in.

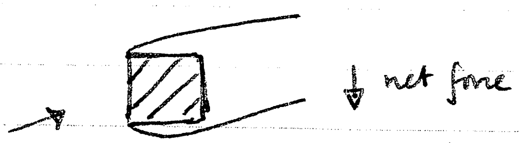
ii) Wind from direction B - motion will be in the towers as before, but the walkway will also gallop.

(1 mark)

• Gallop arises because the vertical motion of the walkway changes the apparent angle α of attack in the wind. This change in α causes a change in the vertical fluid force on the walkway. For a square cross-section there is a downwards force when the walkway is moving downwards (1 mark)

• This arises due to boundary layer separation:





• The downwards force is proportional to velocity and can exceed the damping force. If this happens, the walkway will be unstable to soft excitation. (1 mark)

• Galloping can be prevented by ensuring that $\partial C_y/\partial\alpha|_{\alpha=0}$ is negative (although this may not prevent galloping due to hard excitation) or increasing ω_y, ζ or m so that the velocity of onset of galloping exceeds normal velocities encountered by the walkway. (1 mark)

+ 1 discretionary mark for excellent answer

b) $$m\ddot{y} + 2m\zeta\omega_n\dot{y} + ky = C_y\left(\tfrac{1}{2}\rho U^2 D\right)$$

C_y depends on α, the apparent angle of attack. Soft excitation starts from $\alpha = 0$ so we can make the approximation that:

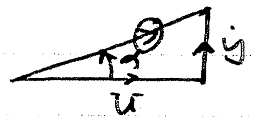
(1 mark) $$C_y \simeq C_y(\alpha=0) + \alpha \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0}$$

~~If we~~ The $C_y(\alpha=0)$ component of the force (if any exists) can be cancelled out by a structural force, so we are entitled to say:

$$C_y \simeq \alpha \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0}$$

Furthermore, the ~~relative~~ apparent angle of attack α is given by $\alpha = \dot{y}/U$:

(1 mark)



← frame of reference moving with walkway (downwards)

hence: $$m\ddot{y} + 2m\zeta\omega_n\dot{y} + ky = \frac{1}{2}\rho U^2 D \frac{\dot{y}}{U} \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0}$$

(2 marks) $$\Rightarrow m\ddot{y} + \left(2m\zeta\omega_n - \frac{1}{2}\rho U D \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0}\right)\dot{y} + ky = 0$$

This will be unstable when the damping term is negative, i.e. when:

(2 marks) $$\frac{1}{2}\rho U D \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0} > 2m\zeta\omega_n$$

c) From the graph, $$\left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0} \simeq \frac{0.3}{5 \times \frac{\pi}{180}} = 3.44$$ { 1 mark for 3.0 → 3.9, 0 otherwise (or 1 mark)

$$\Rightarrow U_{crit} = \frac{2m\zeta\omega_n}{\frac{1}{2}\rho D \left.\partial C_y/\partial\alpha\right|_0} = \frac{2 \times 1000 \times 0.1 \times 0.5}{\frac{1}{2} \times 1.3 \times 3 \times 3.44} = 14.9\,\mathrm{ms^{-1}}$$

$$= 53.7\ \mathrm{km\,h^{-1}}$$

d) If the damping term is cancelling to zero, $m\ddot{y} + ky = 0$. This has ~~resonant~~ frequency $\omega_n = \sqrt{\frac{k}{m}}$, which of course is the resonant frequency of the walkway. $= 0.5\ \mathrm{rad\,s^{-1}}$ (2 marks)