CRIB: 4C1 – 2004

1. (a) Line tension T is the force acting along the dislocation line, which equals the energy per unit length of the dislocation:

$$T \sim Gb^2/2$$

To minimise the energy (and hence the tension T), the dislocation tends to be straight in the absence of stresses. (15%)

(b) After bowing, a segment of length dx of the dislocation has the new length $\sqrt{1+u'^2}\,dx$, with $u' \equiv du/dx$. The total length of dislocation is $\int_{-\ell/2}^{\ell/2} \sqrt{1+u'^2}\,dx$, and the total energy due to line tension is

$$\int_{-\ell/2}^{\ell/2} T\sqrt{1+u'^2}\,dx$$

On the other hand, during bowing, work is done against resistance τb, namely

$$\int_{-\ell/2}^{\ell/2} \tau b\, u(x)\,dx$$

(25%)

$\Rightarrow$ Total energy of the bowed segment is

$$E = \int_{-\ell/2}^{\ell/2} T\sqrt{1+u'^2}\,dx - \int_{-\ell/2}^{\ell/2} \tau b\, u(x)\,dx$$

(c) The configuration of the bowed-out segment that minimises the energy is obtained by solving:

$$\frac{d}{dx}\left[\frac{Tu'}{\sqrt{1+u'^2}}\right] = -\tau b$$

$$\Rightarrow \quad \frac{Tu'}{\sqrt{1+u'^2}} = -\tau b x + C_1$$

1. (c) continued

$$\Rightarrow \quad \frac{u'^2}{1+u'^2} = \left(\frac{\tau b x}{T} + c\right)^2$$

From this, it can be shown that

$$\frac{du}{dx} = \pm \frac{\frac{\tau b x}{T} + c}{\sqrt{1-\left(\frac{\tau b x}{T}+c\right)^2}} \qquad c = 0 \; . \; \frac{du}{dx} = 0 \text{ at } x = 0$$

Choose "$-$" because $\frac{du}{dx} < 0$ when $x > 0$

Integration

$$\Rightarrow \quad u(x) = -\int \frac{\tau b x/T}{\sqrt{1-(\tau b x/T)^2}}\,dx + d$$

$$= \frac{T}{\tau b}\sqrt{1-\left(\frac{\tau b x}{T}\right)^2} + d = \sqrt{\left(\frac{T}{\tau b}\right)^2 - x^2} + d$$

But $u(\pm \frac{l}{2}) = 0 \quad \Rightarrow \quad d = -\sqrt{\left(\frac{T}{\tau b}\right)^2 - \frac{l^2}{4}}$ (35%)

$$\Rightarrow \quad u(x) = \sqrt{\left(\frac{T}{\tau b}\right)^2 - x^2} - \sqrt{\left(\frac{T}{\tau b}\right)^2 - l^2/4}$$

(d) When $|u'| \ll 1$, the differential equation simplifies to

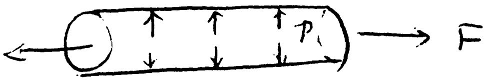
$$T\frac{du'}{dx} = -\tau b \quad \text{or} \quad \frac{d^2u}{dx^2} = -\frac{\tau b}{T}$$

integration

$$\Rightarrow \quad u(x) = -\frac{\tau b}{2T}x^2 + c_1 x + d_1$$ (15%)

But $u(\pm\frac{l}{2}) = 0 \quad \Rightarrow \quad c_1 = 0, \; d_1 = \frac{\tau b}{2T}\frac{l^2}{4}$

$$\Rightarrow \quad u(x) = \frac{\tau b}{2T}\left(\frac{l^2}{4} - x^2\right)$$

2. (a) 

$$\sigma_H = \frac{PR}{t}, \quad \sigma_L = \frac{F}{2\pi R t}, \quad \sigma_T = 0$$

If $\sigma_H > \sigma_L$, the ordering of principal stresses is $\sigma_1 = \sigma_H$, $\sigma_2 = \sigma_L$, $\sigma_3 = 0$

If $\sigma_H < \sigma_L$, - - - - $\sigma_1 = \sigma_L$, $\sigma_2 = \sigma_H$, $\sigma_3 = 0$

mean stress $= (\sigma_1 + \sigma_2 + \sigma_3)/3$

For aluminium $\dot{\varepsilon}_1 = \lambda \sigma_1' = \lambda(\sigma_1 - \sigma_m)$, $\dot{\varepsilon}_2 = \lambda(\sigma_2 - \sigma_m)$, $\dot{\varepsilon}_3 = \lambda(\sigma_3 - \sigma_m)$

$\Rightarrow (\dot{\varepsilon}_1 - \dot{\varepsilon}_2) = \lambda(\sigma_1 - \sigma_2)$ and so on

$$\Rightarrow \dot{\varepsilon}_e = \sqrt{\frac{2}{9}\left[(\dot{\varepsilon}_1 - \dot{\varepsilon}_2)^2 + (\dot{\varepsilon}_2 - \dot{\varepsilon}_3)^2 + (\dot{\varepsilon}_3 - \dot{\varepsilon}_1)^2\right]}$$

$$= \sqrt{\frac{2}{9}\lambda^2\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} = \frac{2}{3}\lambda\sigma_e$$

But $\dot{\varepsilon}_e = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0}\right)^6 \Rightarrow \lambda = \frac{3}{2}\frac{\dot{\varepsilon}_0}{\sigma_0}\left(\frac{\sigma_e}{\sigma_0}\right)^5$ (40%)

$$\Rightarrow \dot{\varepsilon}_1 = \frac{3}{2}\frac{\dot{\varepsilon}_0}{\sigma_0}\left(\frac{\sigma_e}{\sigma_0}\right)^5(\sigma_1 - \sigma_m), \quad \dot{\varepsilon}_2 = \cdots, \quad \dot{\varepsilon}_3 = \cdots$$

(b) For copper

Assume $\sigma_H > \sigma_L$

$\sigma_1 = \sigma_H = \frac{PR}{t}$, $t_f^{Cu} = A\sigma_H^{-6}$ (swap σ_H with σ_L if $\sigma_H < \sigma_L$)

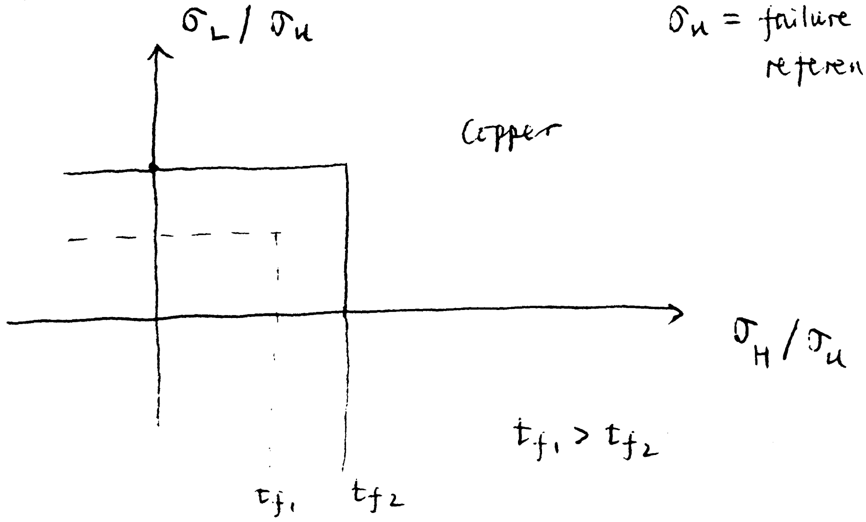
For aluminium alloy

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} = \sqrt{\frac{1}{2}\left[\sigma_H^2 + \sigma_L^2 + (\sigma_H - \sigma_L)^2\right]}$$

$$t_f^{AL} = A\sigma_e^{-6} \qquad = \frac{\sigma_H}{\sqrt{2}}\left[1 + \alpha^2 + (1-\alpha)^2\right]^{1/2}, \quad \alpha = \frac{\sigma_L}{\sigma_H}$$

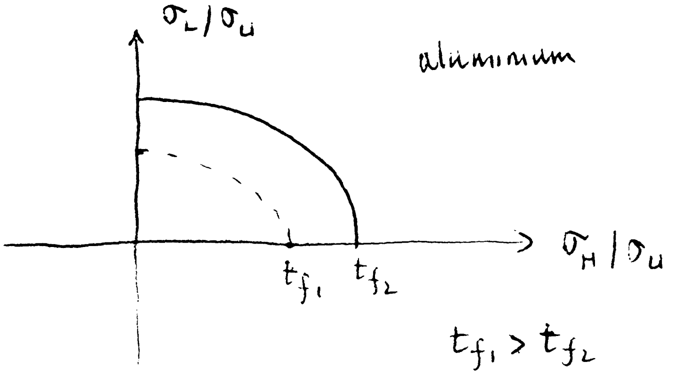
σ_u = failure stress at a reference temp

(25%)



$t_{f1} > t_{f2}$

2. (b) continued



$t_{f_1} > t_{f_2}$

(c) If $\sigma_H > \sigma_L \Rightarrow \alpha < 1$

$$\Rightarrow \quad I = \frac{t_f^{Cu}}{t_f^{Al}} = \left(\frac{\sigma_e}{\sigma_H}\right)^6 = \left\{\frac{1}{2}\left[1+\alpha^2+(1-\alpha)^2\right]\right\}^3$$

If $I = 1$, $\quad 1+\alpha^2+(1-\alpha)^2 = 2 \quad \Rightarrow \alpha(\alpha-1)=0$, $\quad \alpha = 0$ or 1

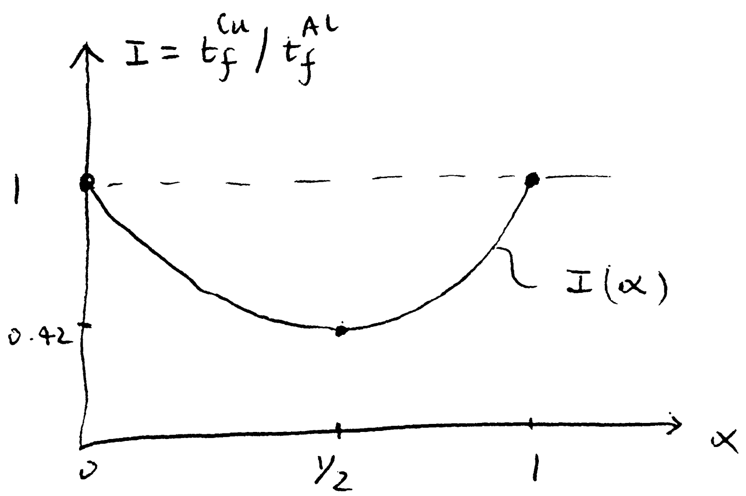
At minimum of I, $\quad \frac{dI}{d\alpha} = 0$

$$\Rightarrow \quad 2\alpha - 2(1-\alpha) = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2}$$

$$\Rightarrow \quad t_f^{Cu}/t_f^{Al} = \begin{cases} 1, & \alpha = 0 \quad (\sigma_L = 0) \\ 1, & \alpha = 1 \quad (\sigma_L = \sigma_H) \\ \text{minimum}, & \alpha = \frac{1}{2} \quad (\sigma_L = 2\sigma_H) \end{cases}$$

(35%)

$I_{minimum} = 0.42$

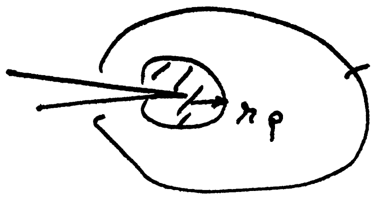


Note that at $I = I_{minimum}$, $\sigma_L = 2\sigma_H$, which corresponds to the case of internally pressurized cylinderical vessel with closed ends.

Q3

(a)

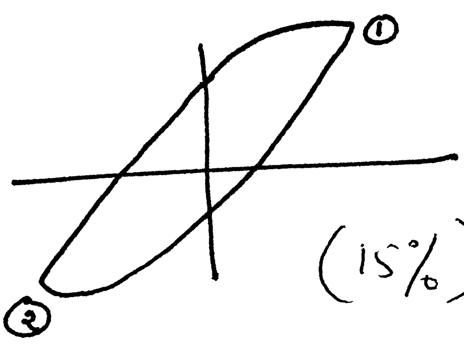
Small scale yielding : In small scale yielding, the plastic zone size $r_p \sim \frac{K^2}{\pi \sigma_Y^2}$. This is much less than the leading dimension like plate size L, crack length a etc.



In SSY, the non-linear zone is completely embedded within an outer elastic K-field which determines the state in the non-linear zone & thus K is an adequate parameter to correlate fracture.

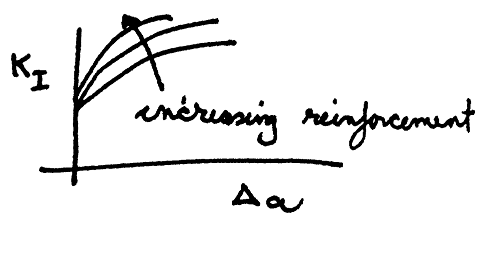
In large scale yielding, the plastic zone is of the order of the leading dimension. No ~~K~~ K-field exists but J might be used to correlate fracture. (20%)

(b) Hysteresis in the stress versus strain curve is the origin of R-curves in metals



(15%)

The R-curve effect increases with fibre re-inforcement ie.



due to more crack bridging by the fibres in the wake of the crack.

c

(i) $$K_I = \frac{2P}{\sqrt{2\pi L}} \quad \& \quad \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \quad @ \ \theta = 0°$$

By the Irwin criterion (for plane stress) yielding occurs when

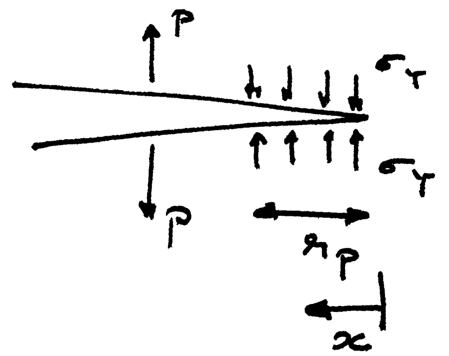
$$\sigma_{yy} = \sigma_Y$$

ie $$r_p = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_Y}\right)^2$$

(15%)

$$r_p = \frac{L}{\pi^2}\left(\frac{P}{\sigma_Y L}\right)^2$$

(ii)



$$K_I^{(P)} \text{ at fictitious crack tip} = \frac{2P}{\sqrt{2\pi(L+r_p)}}$$

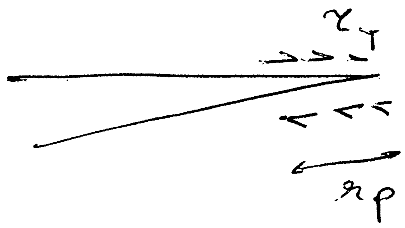
(25%)

$$K_I^{(\sigma_Y)} = -\int_0^{r_p} \frac{2\sigma_Y}{\sqrt{2\pi x}}\,dx = -\frac{4\sigma_Y r_p^{\frac{1}{2}}}{\sqrt{2\pi}}$$

$$K_I = K_I^{(P)} + K_I^{(\sigma_Y)} = 0$$

ie $$r_p = \frac{L}{2}\left[\left\{1 + \left(\frac{P}{L\sigma_Y}\right)^2\right\}^{\frac{1}{2}} - 1\right]$$

(d)

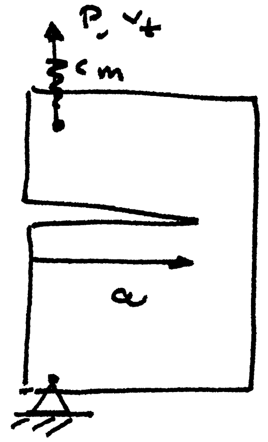


$$K_{II} = \int_0^{r_p} \frac{2\tau_Y}{\sqrt{2\pi x}}\, dx$$

$$= \tau_Y \sqrt{\frac{2}{\pi}}\; 2\, r_p^{1/2}$$

$$r_p = \frac{\pi}{8}\left(\frac{K_{II}}{\tau_Y}\right)^2 \qquad (25\%)$$

4 (a)



$v = c(a)\,P$

compliance of cracked body

$v_t = v + c_m P$

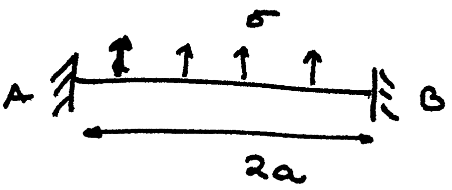
$$E = \frac{1}{2}Pv + \frac{1}{2}P(v_t - v)$$

$$= \frac{1}{2}\frac{v^2}{c} + \frac{1}{2}\frac{(v_t - v)^2}{c_m}$$

$$G = \frac{-\partial E}{B\,\partial a} = \frac{1}{2B}\left[\frac{v^2}{c^2}\frac{dc}{da} \underbrace{- \frac{v}{c}\frac{dv}{da} + \frac{v_t - v}{c_m}\frac{dv}{da}}_{=0 \ \because\ P = \frac{v}{c} = \frac{v_t - v}{c_m}}\right]$$

ie $G = \dfrac{1}{2B} P^2 \dfrac{dc}{da}$ (30%)

(b) (i)



bending moment $M_A = \dfrac{-\sigma B a^2}{3}$

ie $M = \dfrac{-\sigma B}{6}\left(2a^2 - 6ax + 3x^2\right)$

$0 \le x \le a$

$$U = 4\int_0^a \frac{M^2 dx}{2EI} = \frac{8}{15}\frac{\sigma^2 a^5 B}{E h^3} \qquad \left(\because I = \frac{Bh^3}{12}\right)$$

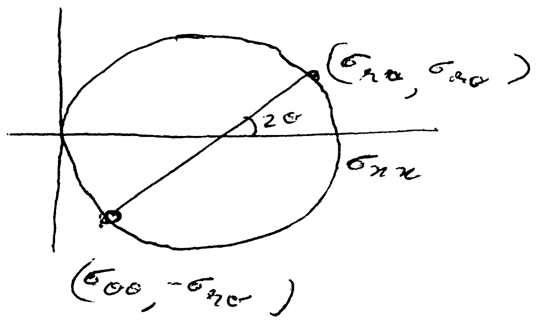
$G = \dfrac{\partial U}{\partial A}$ where $A = 2aB$ (+ sign $\because$ constant load)

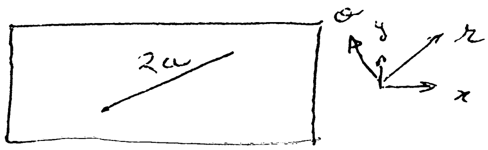
$$G = \frac{4\sigma^2 a^4}{3 E h^3}$$ (40%)

(ii) $K_I = \sqrt{EG}$ (plane stress)

$$= \frac{2\sigma a^2}{\sqrt{3h^3}}$$ (10%)

(c) $\sigma_{xx} = \frac{F}{2\pi Rt}$, $\sigma_{yy} = 0$, $\sigma_{xy} = 0$





$$\sigma_{\theta\theta} = \frac{F}{4\pi Rt}(1 - \cos 2\theta)$$

$$\sigma_{r\theta} = \frac{F}{4\pi Rt} \sin 2\theta$$ (20%)

$$K_I = \frac{F}{4\pi Rt}(1 - \cos 2\theta)\sqrt{\pi a}$$

$$K_{II} = \frac{F}{4\pi Rt} \sin 2\theta$$