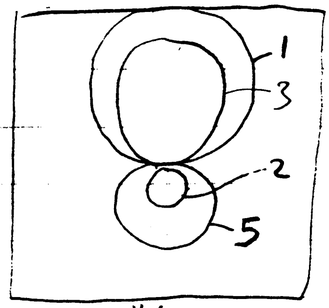
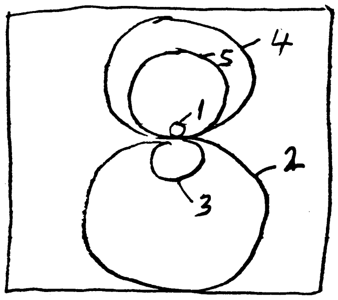
ENGINEERING TRIPOS PART IIB 2004 MODULE 4C6

1 (a)(i)



#1

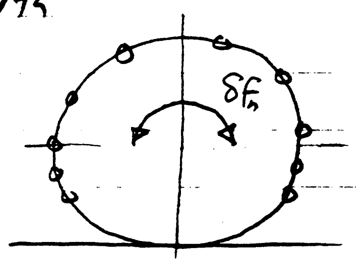


#2

identify circles by comparing with peak height on the transfer functions [10%]

(ii) For Q_n use half power bandwidth

$$Q_n = \frac{\omega_n}{\delta\omega_n} = \frac{f_n}{\delta f_n}$$



δf_n is found by counting the number of dots and multiplying by 0.0488 Hz

Mode	f_n	number of dots (N)	$\delta f_n = N \times 0.0488$	$Q_n = \frac{f_n}{\delta f_n}$
1	10	8	0.39	26
2	20	8	0.39	51
3	45	9	0.44	102
4	70	14.5	0.71	99
5	80	8	0.39	205

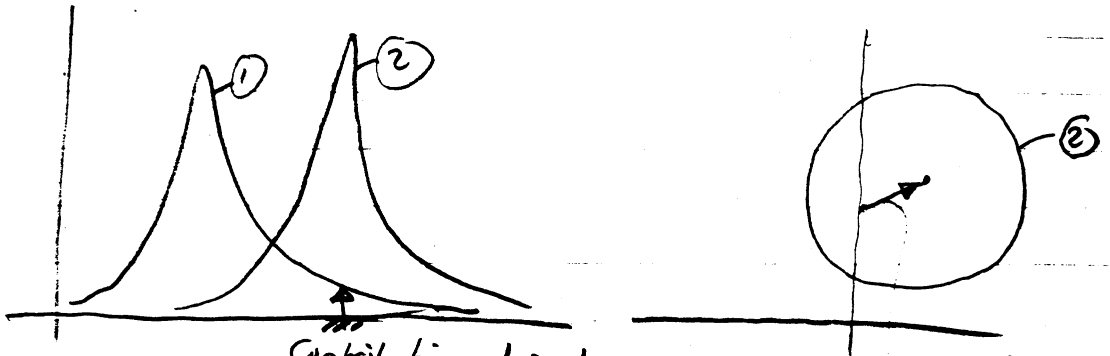
[40%]

(iii) For $U_n(x)\,U_n(y)$ use peak value $= i\,Q_n\,U_n(x)\,U_n(y)$

Mode	Q_n	peak #1	peak #2	$U_n(x)U_n(y)$ #1	$U_n(x)U_n(y)$ #2
1	26	1.0	0.1	0.04	0.004
2	51	−0.2	−1.0	−0.004	−0.02
3	102	0.8	−0.2	0.008	−0.002
4	99	0	0.8	0	0.008
5	205	−0.5	0.5	−0.0025	0.0025

[30%]

1 cont. (b) (i) The centre of modal circles are shifted if the adjacent modes are contributing

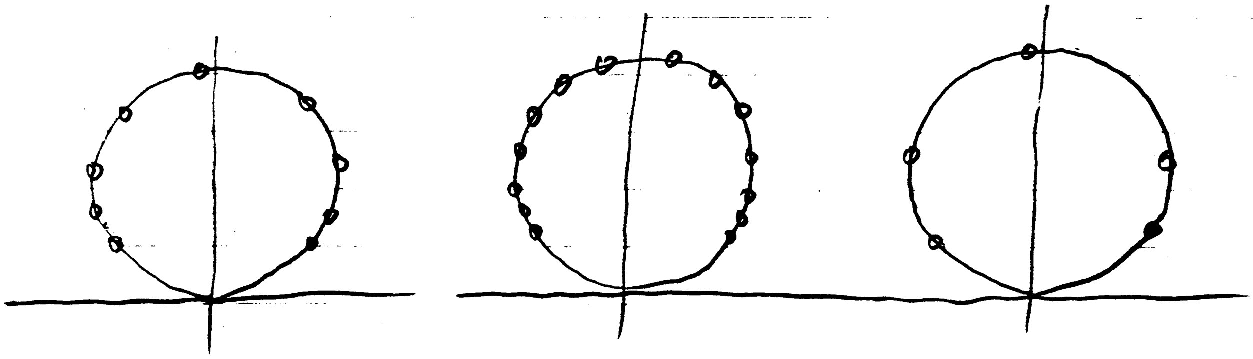


Contribution due to mode 1 is roughly constant

centre offset

[10%]

(ii) Points are closer together in modal circles if the total number of sampled data points is increased but they are further apart if the sampling rate is increased.



$N_s = 4096$ $f_s = 200$

$N_s = 8192$ $f_s = 200$

$N_s = 4096$ $f_s = 400$

This is because the number of points on the circle goes as $\frac{1}{\Delta f}$ where Δf is the frequency bin width $\Delta f = \frac{f_{sample}}{N_{samples}}$

[10%]

2 a) (i) $\omega_1 = \omega_2 = \sqrt{\frac{k}{m}}$ $\quad \underset{\sim}{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \underset{\sim}{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ [10%]

(ii) Equations of motion:

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k+k_c & -k_c \\ -k_c & k+k_c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For natural frequencies $|-\omega^2 M + k| = 0$, so:-

$$\begin{vmatrix} -\omega^2 M + k + k_c & -k_c \\ -k_c & -\omega^2 M + k + k_c \end{vmatrix} = 0$$

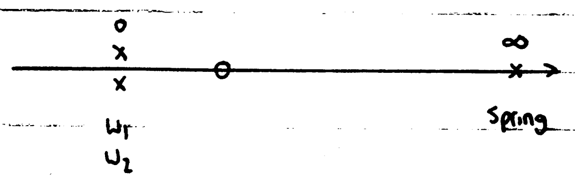
$$\Rightarrow (-\omega^2 M + k + k_c)^2 = k_c^2 \Rightarrow -\omega^2 M + k + k_c = \pm k_c$$

$$\Rightarrow \underline{\omega_1 = \sqrt{\frac{k}{M}}} \ ; \ \underline{\omega_2 = \sqrt{\frac{k+2k_c}{M}}}$$

For mode 1 shape $(-\omega^2 M + k + k_c)x_1 - k_c x_2 = 0 \Rightarrow k_c x_1 - k_c x_2 = 0$ $\quad \underset{\sim}{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For mode 2 shape $(-\omega^2 m + k + k_c)x_1 - k_c x_2 = 0 \Rightarrow -k_c x_1 - k_c x_2 = 0$ $\quad \underset{\sim}{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

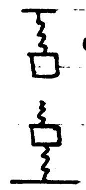
For interlacing:



x before coupling
o after coupling

After coupling ω_n's (o) interlace with those before coupling (x)

NB, can consider uncoupled system as:



$\omega_n = \sqrt{k/M}$

$\omega_n = \sqrt{k/M}$ and ∞

[40%]

2 cont.

b) (i) $\frac{c^2}{R^2}\frac{\partial^2 p}{\partial\theta^2} - \ddot{p} = 0$

p harmonic $\Rightarrow \ddot{p} = -\omega^2 p \Rightarrow \frac{\partial^2 p}{\partial\theta^2} + \frac{R^2\omega^2}{c^2} p = 0$

$\Rightarrow p = \cos\left(\frac{R\omega}{c}\right)\theta$ or $\sin\left(\frac{R\omega}{c}\right)\theta$

Must have $p(\theta) = p(\theta + 2\pi n)$ for physical result $\Rightarrow 2\pi\left(\frac{R\omega}{c}\right) = 2\pi n \Rightarrow \underline{\omega_n = nc/R}$

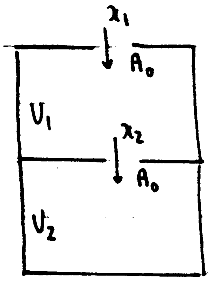
Mode shapes:

Mode shapes:	$n=0$	$p = 1$
	$n=1$	$p = \cos\theta$ <u>and</u> $\sin\theta$
	$n=2$	$p = \cos 2\theta$ <u>and</u> $\sin 2\theta$ etc

[30%]

(ii) Effects of small hole: adds some <u>mass</u>, similar to the hole in a Helmholtz resonator. Also adds <u>damping</u> due to viscous air flow. Both effects act ~~to~~ separate the pair of modes which were previously degenerate.
First two non-zero frequencies were the pair of modes for $n=1$. With the hole, there will be one mode with a pressure node at the hole, and another with an antinode there. The one with the node will be exactly as before, but the one with the antinode will have its frequency reduced (by the added mass) and its damping factor increased.

3 a)



Degrees of Freedom : "neck" mass displacements x_1 and x_2

Hole area : A_0

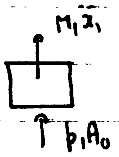
Change in density for V_2 : $\left(\frac{A_0 x_2}{V_2}\right)\rho$

Increase in pressure for V_2 : $c^2\rho\,(A_0 x_2/V_2) = p_2$

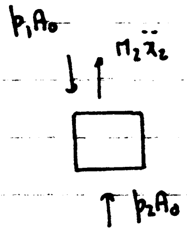
Change in density for V_1 : $\frac{A_0(x_1-x_2)\rho}{V_1}$

Increase in pressure for V_1 : $c^2\rho A_0(x_1-x_2)/V_1 = p_1$

Equation of motion for top mass:



$$M_1\ddot{x}_1 + p_1A_0 = 0 \Rightarrow M_1\ddot{x}_1 + \frac{c^2\rho A_0^2}{V_1}(x_1-x_2) = 0$$



$$M_2\ddot{x}_2 + A_0(p_2-p_1) = 0 \Rightarrow M_2\ddot{x}_2 + c^2\rho A_0^2\left[x_2\left(\frac{1}{V_1}+\frac{1}{V_2}\right) - \frac{x_1}{V_1}\right] = 0$$

$$\Rightarrow \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + c^2\rho A_0^2 \begin{pmatrix} \frac{1}{V_1} & -\frac{1}{V_1} \\ -\frac{1}{V_1} & \frac{1}{V_1}+\frac{1}{V_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

[30%]

b) $V_1 = b\pi R^2(1-\alpha)$

$V_2 = b\pi R^2\alpha$

$A_0 = \pi a^2$

$M_1 = M_2 = \pi a^2\rho L = 1.7\pi\rho a^3$

effective neck length $= 1.7a$, from notes

$$\Rightarrow 1.7\pi\rho a^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \frac{c^2\rho(\pi a^2)^2}{b\pi R^2\alpha(1-\alpha)}\begin{pmatrix} \alpha & -\alpha \\ -\alpha & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \frac{c^2 a}{1.7R^2\alpha(1-\alpha)\,b}\begin{pmatrix} \alpha & -\alpha \\ -\alpha & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

call this β

5

3 cont.

For natural frequencies
$$\begin{vmatrix} -\omega^2+\alpha\beta & -\alpha\beta \\ -\alpha\beta & -\omega^2+\beta \end{vmatrix} = 0$$

$$(-\omega^2+\alpha\beta)(-\omega^2+\beta) - \alpha^2\beta^2 = 0$$
$$\omega^4 - \omega^2(\alpha\beta+\beta) - \alpha^2\beta^2 + \alpha\beta^2 = 0$$

$$2\omega^2 = (\alpha\beta+\beta) \pm \sqrt{\alpha^2\beta^2+\beta^2+2\alpha\beta^2+4\alpha^2\beta^2-4\alpha\beta^2}$$

$$= (\alpha\beta+\beta) \pm \sqrt{\beta^2(5\alpha^2-2\alpha+1)}$$

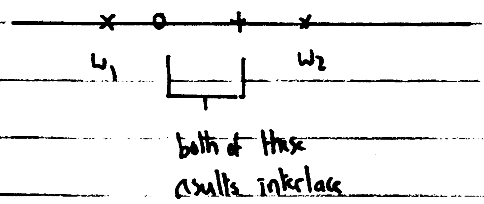
$$\underline{\omega^2 = \frac{\beta}{2}\left\{(\alpha+1) \pm \sqrt{(5\alpha^2-2\alpha+1)}\right\}}$$

Lower hole blocked $\Rightarrow x_2 = 0 \Rightarrow \omega^2 = \alpha\beta$

Upper hole blocked $\Rightarrow x_1 = 0 \Rightarrow \omega^2 = \beta$

Coupled system $\omega^2 = \frac{\beta}{2}\left\{(\alpha+1) \pm \sqrt{(\alpha-1)^2+4\alpha^2}\right\}$ + root $> \beta$

− root $< \alpha\beta$



x coupled

o lower blocked

+ upper blocked

[60%]

c) $\omega_1^2 = \frac{\beta}{2}\left\{(\alpha+1) - \sqrt{(\alpha-1)^2+4\alpha^2}\right\}$ $\beta = \frac{c^2 a}{1.7R^2\alpha(1-\alpha)b}$

For $\alpha \to 1$ $\omega_1^2 \simeq \frac{\beta}{2}\{\alpha+1-2\alpha\} = \frac{\beta}{2}(1-\alpha) = \underline{\frac{c^2a}{2\times 1.7R^2b}}$

This is the natural frequency of a single volume with neck mass $= 2\times 1.7\,\pi\rho a^3$

Physical result would have neck mass $= 1.7\pi\rho a^3 \Rightarrow$ erroneous factor of 2 because "interaction" of neck masses and the effect of a shallow cavity V_1 neglected. [30%]

4 (a) Potential energy $V = \frac{1}{2}kx^2 + \frac{1}{2}\lambda k y^2$

$$\therefore \quad K = \begin{bmatrix} k & 0 \\ 0 & \lambda k \end{bmatrix}$$

Kinetic energy $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\lambda m(\dot{x}+\dot{y})^2$

$$\therefore \quad M = \begin{bmatrix} m+\lambda m & \lambda m \\ \lambda m & \lambda m \end{bmatrix}$$

[10%]

(b) For $\lambda \ll 1$, mode are approximately $\underline{u} = \begin{bmatrix} \sqrt{\lambda} \\ \pm 1 \end{bmatrix}$.

Can get frequencies either from Rayleigh or from equations of motion. Use Rayleigh:

$$\omega^2 \simeq \frac{\underline{u}^t K \underline{u}}{\underline{u}^t M \underline{u}}$$

$$= \frac{k\lambda + k\lambda}{m\lambda + \lambda m(\sqrt{\lambda} \pm 1)^2} = \frac{2k}{m}\left[1 + \lambda \pm 2\sqrt{\lambda} + 1\right]^{-1}$$

$$\simeq \frac{k}{m}(1 \pm \sqrt{\lambda})^{-1} \simeq \frac{k}{m}(1 \mp \sqrt{\lambda})$$

[20%]

(c) Now add damping: Rayleigh quotient is

$$R = \frac{kx^2 + \lambda k(1+i\eta)y^2}{(m+\lambda m)x^2 + \lambda m y^2 + 2\lambda m xy}$$

Given a reasonable approximation to a mode shape, this will give an estimate of the complex ω^2. Provided damping is small, the undamped modes give an approximation to the damped modes. So use previous modes:

$$\omega^2 \simeq \frac{k\lambda + \lambda k(1+i\eta)}{(m+\lambda m)\lambda + \lambda m \pm 2\lambda m\sqrt{\lambda}}$$

$$= \frac{k}{m} \frac{2+i\eta}{2+\lambda \pm 2\sqrt{\lambda}} \quad (1)$$

4 cont.

Complex frequency $\omega = \omega_0\left(1 + \frac{i}{2Q}\right)$

So $\frac{1}{2Q} = \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}$

But for small damping, $\omega^2 \simeq \omega_0^2\left(1 + \frac{i}{Q}\right)$

So simpler formula is $\frac{1}{Q} \simeq \frac{\mathrm{Im}(\omega^2)}{\mathrm{Re}(\omega^2)}$

So use (1) : $\frac{1}{Q} \simeq \frac{\eta}{2}$

i.e. same damping factor for both modes, with half the damping of the absorber on its own. [35%]

(d) Tuned absorber only hits a narrow frequency range, so suitable when only one mode is causing a problem, usually when excitation is narrow band, sometimes hitting a natural frequency. Examples are rotating machines or reciprocating ones, e.g. hammer drills, orbital sanders, hair clippers. If the casing or internal equipment has a resonance near the operating frequency, a tuned absorber would be worth trying. Design limits: the smaller λ, the more compact the design, but the larger the movement of the absorber mass (factor $\sqrt{2\lambda}$). So there is conflict between size/mass, and allowing a big enough travel to the absorber mass. Also want to add a lot of damping, but (i) hard to find suitable lossy spring material, and (ii) the approximation used here relies on small damping, so if damping is too high, need a more exact analysis using first-order method. [35%]