4C7 2004 Cribs

1. a) $M\ddot{x} + C\dot{x} + kx = F(t)$

$\downarrow$

white noise with spectrum S_0

Standard results $\sigma_x^2 = \frac{\pi S_0}{Ck} = \frac{\pi \times 1}{0.5 \times 18} = 0.349 \Rightarrow \underline{\sigma_x = 0.59\ \text{m}}$

$\sigma_{\dot{x}}^2 = \frac{\pi S_0}{MC} = \frac{\pi \times 1}{2 \times 0.5} = 3.142 \Rightarrow \underline{\sigma_{\dot{x}} = 1.77\ \text{m/s}}$

$\sigma_F = k\,\sigma_x = \underline{10.62\ \text{N}}$

$\sigma_{\dot{F}} = k\,\sigma_{\dot{x}} = \underline{31.86\ \text{N/s}}$

$\nu_0^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{F}}}{\sigma_F}\right) = \underline{0.477\ s^{-1}}$

Mean peak height $= \sqrt{\frac{\pi}{2}}\,\sigma_F = \underline{13.3\ \text{N}}$

} formulae from notes [30%]

b) Failure can arise from (i) exceeding the level 3.2m, (ii) causing fatigue failure of the spring.

For (i) $\nu_b^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) e^{-\frac{1}{2}(b/\sigma_x)^2}$ with $b = 3.2$

$\Rightarrow \nu_b^+ = \frac{1}{2\pi}\left(\frac{1.77}{0.59}\right) e^{-\frac{1}{2}(b/0.59)^2} = 1.95\times10^{-7}$

Probability of crossing $= 1 - e^{-\nu_b^+ T} = 1 - e^{-1.95\times10^{-7}\times 3\times60\times60} = \underline{0.0021} < 0.01$

For (ii) Fatigue damage $= E[D] = \nu_0^+ T \int_0^\infty \frac{1}{N(b)}\, p(b)\, db \times 5$ ← safety factor

$\frac{1}{N(b)}$: $2\times10^{-5}\, b$ from question

$p(b)$: $\frac{b}{\sigma_F^2}\, e^{-\frac{1}{2}(b/\sigma_F)^2}$ from notes

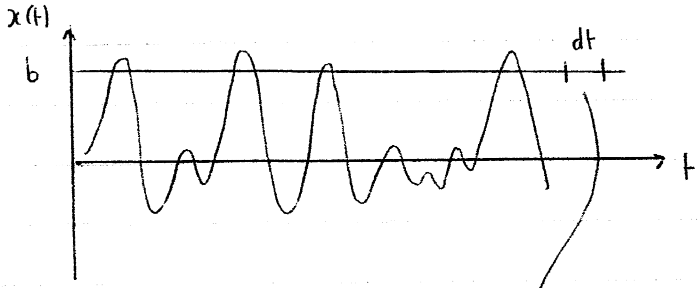
$$E[D] = 5 \times 0.477 \times 3\times60\times60 \times 2\times10^{-5} \underbrace{\int_0^\infty \frac{b^2}{\sigma_F^2}\, e^{-\frac{1}{2}(b/\sigma_F)^2}\, db}_{\frac{1}{2}\sqrt{2\pi}\,\sigma_F} = \underline{6.85 > 1}$$

$\Rightarrow$ Structure is unsafe from fatigue point of view [40%]

c) Duration required to cause fatigue damage $= \frac{3}{6.85} = \underline{0.437 \text{ hours}}$

Note the structure will meet the 'probability of crossing' requirement over this reduced duration [30%]

2 a)



To cross in a time interval dt with velocity $> u$, must have

(i) $\dot{x} > u$

(ii) $b - \dot{x}\,dt < x < b$

$\therefore$ Probability of crossing in interval dt $= \int_{b-\dot{x}dt}^{b} \int_{u}^{\infty} p(x,\dot{x})\, dx\, d\dot{x}$

$$P \simeq \int_u^\infty \dot{x}\, p(b,\dot{x})\, d\dot{x}\, dt$$

Number of crossings in N intervals dt = $P \times N$

Also equal to $\nu_b N dt$

$\Rightarrow \underline{\nu_b = \int_u^\infty \dot{x}\, p(b,\dot{x})\, d\dot{x}}$ [30%]

b) $p(x,\dot{x}) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} e^{-\frac{1}{2}(x/\sigma_x)^2 - \frac{1}{2}(\dot{x}/\sigma_{\dot{x}})^2}$

$$\int_u^\infty \dot{x}\, p(b,\dot{x})\, d\dot{x} = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} e^{-\frac{1}{2}(b/\sigma_x)^2} \underbrace{\int_u^\infty \dot{x}\, e^{-\frac{1}{2}(\dot{x}/\sigma_{\dot{x}})^2}\, d\dot{x}}_{\sigma_{\dot{x}}^2\, e^{-\frac{1}{2}(u/\sigma_{\dot{x}})^2}}$$

$$\underline{\nu_b = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) e^{-\frac{1}{2}(b/\sigma_x)^2 - \frac{1}{2}(u/\sigma_{\dot{x}})^2}}$$ [35%]

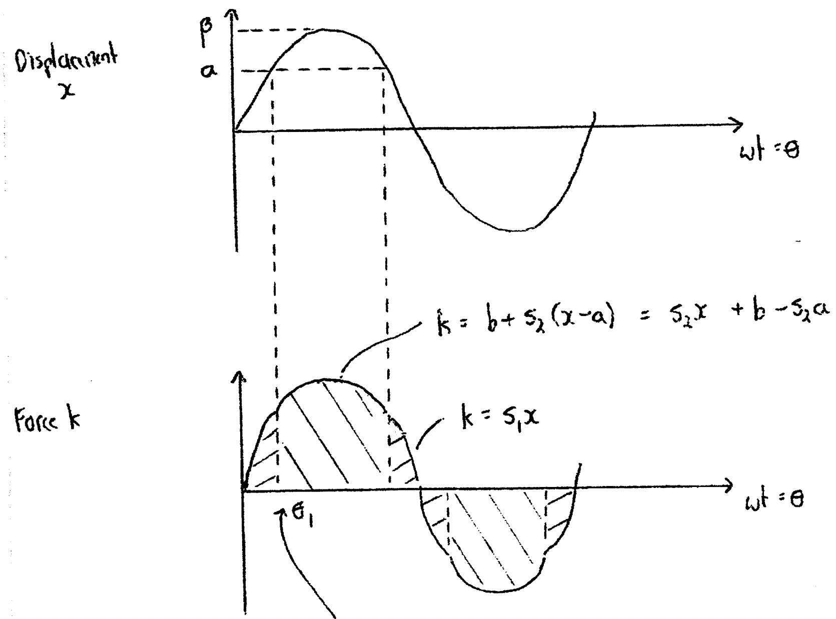
c) Here $b = 14$, $u = 2$, $\sigma_x = 3.75$, $\sigma_{\dot{x}} = 1.5$

$\Rightarrow \nu_b = 0.0636 \times 9.41 \times 10^{-4} \times 0.411 = \underline{2.46 \times 10^{-5}}$ s^{-1}

Assuming crossings are independent, probability of damage $= \underbrace{1 - e^{-\nu_b t}}_{\text{note Poisson Formula!}} = \underline{0.162}$ [35%]

3 a) If $\beta < a$, linear spring case

$\beta > a$, input-output relation



$\beta \sin\theta_1 = a \Rightarrow \theta_1 = \sin^{-1}(a/\beta)$

$$D = \frac{1}{\beta\pi}\int_0^{2\pi} (\text{output}) \sin\theta \, d\theta = \frac{1}{\beta\pi}\left\{ 4\int_0^{\theta_1} (s_1\beta\sin\theta)\sin\theta \, d\theta + 2\int_{\theta_1}^{\pi-\theta_1} (s_2\beta\sin\theta + b - s_2 a)\sin\theta \, d\theta \right\}$$

Now $\int \sin^2\theta \, d\theta = \frac{1}{2}\int (1-\cos 2\theta) d\theta = \frac{1}{2}[\theta - \frac{1}{2}\sin 2\theta]$

and $\int \sin\theta \, d\theta = [-\cos\theta]$

Thus $D = \frac{1}{\beta\pi}\left\{ 2s_1\beta[\theta_1 - \frac{1}{2}\sin 2\theta_1] + s_2\beta[\pi - 2\theta_1 + \sin 2\theta_1] + 4(b - s_2 a)\cos\theta_1 \right\}$

Now $\theta_1 = \sin^{-1}(a/\beta)$; $\sin 2\theta_1 = 2\sin\theta_1\cos\theta_1 = 2(a/\beta)\sqrt{1-(a/\beta)^2}$; $\cos\theta_1 = \sqrt{1-(a/\beta)^2}$

$$\Rightarrow D = \frac{1}{\beta\pi}\left\{ 2s_1\beta\left[\sin^{-1}\left(\frac{a}{\beta}\right) - \left(\frac{a}{\beta}\right)\sqrt{1-\left(\frac{a}{\beta}\right)^2}\right] + s_2\beta\left[\pi - 2\sin^{-1}\left(\frac{a}{\beta}\right) + 2\left(\frac{a}{\beta}\right)\sqrt{1-\left(\frac{a}{\beta}\right)^2}\right] + 4(b-s_2 a)\sqrt{1-\left(\frac{a}{\beta}\right)^2}\right\}$$

For $s_1 = s_2$, $b = s_2 a$, $D \to \frac{1}{\beta\pi}(s_2\beta\pi) = s_2$ ✓ [50%]

b) $m\ddot{x} + Dx = A\sin\omega t$

$$\Rightarrow \underline{(-\omega^2 m + D)\beta = A}$$

↑

substitute for D from previous part of the question [50%]

c) A piecewise linear function could be used to approximate the cubic non-linearity in the Duffing spring force. Actually it would be easier to derive the describing function directly for the cubic non-linearity, rather than the approximation. [10%]

Q4 (a)

$$\ddot{x} + p^2 x + \varepsilon x^2 = a\cos\omega t \quad \text{—(1)}$$

$$x = c_1 \cos\frac{\omega t}{2} + c_2 \cos\omega t \quad \text{—(2)}$$

$$x^2 = c_1^2 \cos\frac{\omega t}{2} + 2c_1c_2\cos\frac{\omega t}{2}\cos\omega t + c_2^2\cos^2\omega t$$

$$= c_1^2\left(\frac{1+\cos\omega t}{2}\right) + 2c_1c_2\left(\frac{\cos\frac{3\omega t}{2} + \cos\frac{\omega t}{2}}{2}\right) + c_2^2\left(\frac{1+\cos 2\omega t}{2}\right).$$

$$= \frac{c_1^2+c_2^2}{2} + c_1c_2\cos\frac{\omega t}{2} + c_1^2\cos\omega t + c_1c_2\cos\frac{3\omega t}{2} + \frac{c_2^2}{2}\cos 2\omega t. \quad \text{—(3)}$$

$$\ddot{x} = -\left(\frac{\omega}{2}\right)^2 c_1\cos\frac{\omega t}{2} - \omega^2 c_2\cos\omega t.$$

From (1)

$$-\left(\frac{\omega}{2}\right)^2 c_1\cos\frac{\omega t}{2} - \omega^2 c_2\cos\omega t + p^2\left(c_1\cos\frac{\omega t}{2} + c_2\cos\omega t\right) + \varepsilon\ (\text{expn. from (3)}) = a\cos\omega t.$$

Equating terms for $\cos\frac{\omega t}{2}$, $\cos\omega t$.

$$-\left(\frac{\omega}{2}\right)^2 c_1 + p^2 c_1 + \varepsilon c_1 c_2 = 0$$

$$-\omega^2 c_2 + p^2 c_2 + \varepsilon \frac{c_1^2}{2} = a$$

(b)

$$x = c_3\cos\omega t + c_4\cos 2\omega t$$

$$x^2 = c_3^2\cos^2\omega t + c_4^2\cos^2 2\omega t + 2c_3c_4\cos\omega t\cos 2\omega t$$

$$= c_3^2\left(\frac{1+\cos 2\omega t}{2}\right) + 2c_3c_4\left(\frac{\cos\omega t + \cos 3\omega t}{2}\right) + c_4^2\left(\frac{1+\cos 4\omega t}{2}\right) \quad \text{—(4)}$$

$$-\omega^2 c_3 \cos\omega t - 4\omega^2 c_4 \cos 2\omega t + p^2 (c_3 \cos\omega t + c_4 \cos 2\omega t)$$
$$+ \varepsilon \left[\text{expn. from } ④ \right] = a \cos\omega t.$$

$$c_3 (p^2 - \omega^2) + \varepsilon c_3 c_4 = a$$
$$c_4 (p^2 - 4\omega^2) + \varepsilon \frac{c_3^2}{2} = 0.$$

(c) Superposition of solutions is not possible for nonlinear systems.