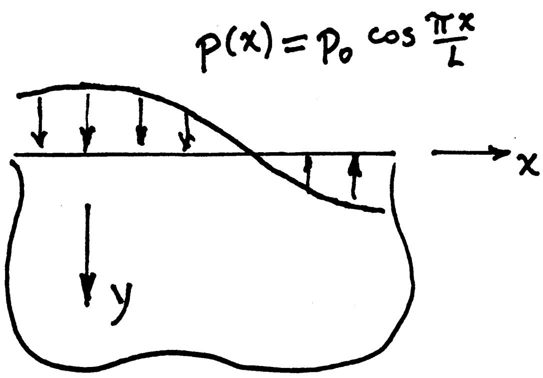
2004 TRIPOS - MODULE 4C9

Q1 (a)i

$$\phi = f(y)\cos\alpha x \quad , \quad \alpha = \pi/L$$



$$\frac{\partial^4\phi}{\partial x^4} = f(y)\,\alpha^4\cos\alpha x$$

$$\frac{\partial^4\phi}{\partial x^2\partial y^2} = -f''(y)\,\alpha^2\cos\alpha x$$

$$\frac{\partial^4\phi}{\partial y^4} = f^{IV}(y)\cos\alpha x$$

$$\therefore \quad 0 = \nabla^2(\nabla^2\Phi) = \left\{ f^{IV}(y) - 2\alpha^2 f''(y) + \alpha^4 f(y)\right\}\cos\alpha x$$

$$0 = (D^2-\alpha^2)^2 f(y)$$

Sol'n:

$$f(y) = Ae^{\alpha y} + Be^{-\alpha y} + Cye^{\alpha y} + Dye^{-\alpha y}$$

(a) ii

Bdy Cond. $\sigma_{yy}(x,\infty) = \sigma_{xy}(x,\infty) = 0 \quad \Rightarrow \quad A = C = 0$

$$\sigma_{yy}(x,y) = \sigma_{yy}(-x,y) \quad \text{requires} \quad \phi(x,y) = \phi(-x,y)$$

$$\sigma_{yy}(x,0) = \frac{\partial^2\phi(x,0)}{\partial x^2} = \alpha^2(B+0)\cos\alpha x = p_o\cos\alpha x$$

$$\sigma_{xy}(x,0) = -\frac{\partial^2\phi(x,0)}{\partial x\partial y} = -\alpha^2(B - D/\alpha)\sin\alpha x = 0$$

$$\therefore \quad B = p_o L^2/\pi^2, \qquad D = p_o L/\pi$$

$$\sigma_{yy}(x,y) = \frac{\partial^2\phi}{\partial x^2} = p_o e^{-\pi y/L}\cos\frac{\pi x}{L}$$

$$\sigma_{xy}(x,y) = -\frac{\partial^2\phi}{\partial x\partial y} = p_o\left[-e^{-\frac{\pi y}{L}} + \left(1-\frac{\pi y}{L}\right)e^{-\frac{\pi y}{L}}\right]\sin\frac{\pi x}{L}$$

Q1 (b)

(i) Problem has circular symmetry; stresses and strains are functions of r only; $\sigma_{r\theta} = 0$.

Equilibrium eqn. $$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

B.C. $\sigma_{rr} = -p$ at $r = a$

$\sigma_{rr} = 0$ at $r = b$

In the yieding zone $a \leq r \leq d$, $\sigma_{rr} - \sigma_{\theta\theta} = -\sigma_0$ (Tresca).

Equilibrium eqn. becomes $$\frac{d\sigma_{rr}}{dr} + \frac{-\sigma_0}{r} = 0$$

$$\Rightarrow \quad \sigma_{rr} = \sigma_0 \ln r + C, \quad \sigma_{\theta\theta} = \sigma_0 (1 + \ln r) + C$$

In the elastic region $d \leq r \leq a$, we have Lame's solutions (Data sheet)

$$\sigma_{rr} = -\frac{A}{r^3} + B, \quad \sigma_{\theta\theta} = \frac{A}{r^3} + B$$

Applying boundary conditions

$$\sigma_{rr} = -p \text{ at } r = a \quad \Rightarrow \quad C = -p - \sigma_0 \ln a$$

$$\Rightarrow \quad \sigma_{rr} = -p + \sigma_0 \ln\frac{r}{a}, \quad \sigma_{\theta\theta} = -p + \sigma_0\left[1 + \ln\frac{r}{a}\right], \quad \sigma_{r\theta} = 0 \quad (a \leq r \leq d)$$

Hence the stress field is statically admissible. With the material exterior to the dashed circle ignored, the above stress field can be used to obtain a lower bound solution for p^L.

p^L is reached when $d = b$. Since $\sigma_{rr} = 0$ at $r = b$ $(= d)$, we obtain

$$p^L = \sigma_0 \ln b/a$$

Q1 (b)

(ii) For plane stress, $\sigma_{zz} = 0$.

When $\ln\frac{b}{a} \le 1$ or $\frac{b}{a} \le e = 2.718$, $\sigma_{\theta\theta} = -p^L + \sigma_0$ at $r = a$ at collapse. Here, $p^L = \sigma_0 \ln\frac{b}{a}$ as given by (i) for plane strain, because $\sigma_{\theta\theta} = +\sigma_0 (1 - \ln\frac{b}{a}) > 0$ so that the yield criterion is still $\sigma_{rr} - \sigma_{\theta\theta} = -\sigma_0$. This is consistent with (i).

When $\ln\frac{b}{a} \ge 1$ or $\frac{b}{a} \ge e$, $\sigma_0 \ln\frac{b}{a} \ge \sigma_0$ is no longer the solution for p^L. This is because if $p \ge \sigma_0$, then at $r = a$ we have $\sigma_{\theta\theta} = -p + \sigma_0 \le 0$. In this case, the proper Tresca's criterion should be

$$\sigma_{rr} - \sigma_{zz} = -\sigma_0 \quad \Rightarrow \quad \sigma_{rr} = -\sigma_0$$

Substitute $\sigma_{rr} = -\sigma_0$ into the equilibrium equation, we have

$$\sigma_{rr} - \sigma_{\theta\theta} = 0 \quad \Rightarrow \quad \sigma_{\theta\theta} = \sigma_{rr} = -\sigma_0$$

The above stress field is valid for the whole cylinder $a \le r \le b$, and hence the corresponding pressure $p^L = \sigma_0$ is the lower bound solution.

In summary, for plane stress

$$p^L = \begin{cases} \sigma_0 \ln\frac{b}{a} & \text{if } \frac{b}{a} \le e \\ \sigma_0 & \text{if } \frac{b}{a} > e \end{cases}$$

Q2

(a) For a thin-walled cylinder subjected to axial tension and torsion, the von Mises yielding criterion is (from Lecture Notes):

$$\sigma_{zz}^2 + 3\sigma_{\theta z}^2 = \sigma_y^2$$

$$\Rightarrow \left(\frac{\sigma_{zz}}{\sigma_y}\right)^2 + \left(\frac{\sigma_{\theta z}}{\sigma_y/\sqrt{3}}\right)^2 = 1 \quad \Rightarrow \quad \sigma^2 + \tau^2 = 1 \qquad (*)$$

On the other hand, Hooke's law $\sigma_{zz} = E\varepsilon_{zz}$, $\sigma_{\theta z} = G\gamma_{\theta z}$ can be rewritten as

$$\sigma \equiv \frac{\sigma_{zz}}{\sigma_y} = \frac{\varepsilon_{zz}}{\sigma_y/E} = \frac{\varepsilon_{zz}}{\varepsilon_y} \equiv \varepsilon$$

$$\tau \equiv \frac{\sigma_{\theta z}}{\tau_y} = \frac{\gamma_{\theta z}}{\tau_y/G} = \frac{\gamma_{\theta z}}{\gamma_y} \equiv \gamma$$

Hence $\sigma^2 + \tau^2 = 1$ can also be written as $\varepsilon^2 + \gamma^2 = 1$. QED.

(b) Levy-Mises relations $\dot{\varepsilon}_{ij}^p = \dot{\lambda} s_{ij}$

$$\Rightarrow \dot{\varepsilon}_{zz}^p = \dot{\lambda} s_{zz} = \dot{\lambda}\frac{2}{3}\sigma_{zz}, \quad \dot{\varepsilon}_{\theta z}^p = \dot{\lambda}\sigma_{\theta z}$$

$$\Rightarrow d\varepsilon_{zz} = \frac{d\sigma_{zz}}{E} + d\lambda \frac{2}{3}\sigma_{zz}, \quad \frac{1}{2}d\gamma_{\theta z} = \frac{d\sigma_{\theta z}}{2G} + d\lambda\,\sigma_{\theta z}$$

Non-dimensionalization gives

$$\frac{d\varepsilon_{zz}}{\varepsilon_y} = \frac{d\sigma_{zz}}{E\varepsilon_y} + \frac{2d\lambda\cdot\sigma_y}{3\varepsilon_y}\cdot\frac{\sigma_{zz}}{\sigma_y} \Rightarrow d\varepsilon = d\sigma + \frac{2Ed\lambda}{3}\sigma$$

$$\frac{d\gamma_{\theta z}}{\gamma_y} = \frac{d\sigma_{\theta z}}{G\gamma_y} + \frac{2d\lambda\cdot\tau_y}{\gamma_y}\cdot\frac{\sigma_{\theta z}}{\tau_y} \Rightarrow d\gamma = d\tau + 2G\,d\lambda\cdot\tau$$

Material incompressible, $\nu = 1/2 \Rightarrow G = \frac{E}{2(1+\nu)} = \frac{E}{3}$

Hence, let $d\lambda' = 2G\,d\lambda$, we have

$$d\varepsilon = d\sigma + \sigma\,d\lambda', \quad d\gamma = d\tau + \tau\,d\lambda'$$

Since $d\lambda' \neq 0$ at yielding, the above can be simplified as $\frac{d\varepsilon - d\sigma}{d\gamma - d\tau} = \frac{\sigma}{\tau}$ (**)

From (*), $\tau = \sqrt{1-\sigma^2}$. Differentiating (*) to obtain $\sigma d\sigma + \tau d\tau = 0$, i.e.

$$d\tau = \frac{-\sigma d\sigma}{\tau} = \frac{-\sigma d\sigma}{\sqrt{1-\sigma^2}}$$

(b) continued.

Substituting $\tau = \sqrt{1-\sigma^2}$ and $d\tau = \frac{-\sigma\, d\sigma}{\sqrt{1-\sigma^2}}$ into (**) gives

$$\frac{d\varepsilon - d\sigma}{d\gamma + \frac{\sigma\, d\sigma}{\sqrt{1-\sigma^2}}} = \frac{\sigma}{\sqrt{1-\sigma^2}}$$

Re-arranging, we obtain $$\frac{d\sigma}{d\varepsilon} = \sqrt{1-\sigma^2}\left(\sqrt{1-\sigma^2} - \sigma\frac{d\gamma}{d\varepsilon}\right) \qquad (***)$$

(c) From $a \to b$, $d\gamma = 0 \Rightarrow \frac{d\sigma}{d\varepsilon} = 1-\sigma^2$

$$\Rightarrow \int_{\varepsilon_0}^{\varepsilon} d\varepsilon = \int_{\sigma_0}^{\sigma} \frac{d\sigma}{1-\sigma^2} \qquad (\sigma < 1 \text{ from } (*))$$

$$\Rightarrow \varepsilon - \varepsilon_0 = \frac{1}{2}\ln\frac{1+\sigma}{1-\sigma} - \frac{1}{2}\ln\frac{1+\sigma_0}{1-\sigma_0}$$

$$\Rightarrow \varepsilon = \varepsilon_0 + \frac{1}{2}\ln\left(\frac{1+\sigma}{1+\sigma}\cdot\frac{1-\sigma_0}{1+\sigma_0}\right)$$

(3) (a) Equilibrium

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho y \Omega^2 = 0$$

try $\tau_{xy} = 0$ so that $\sigma_{yy} = C - \rho \frac{y^2 \Omega^2}{2}$

<u>Bdy Cond.</u>

$$\sigma_{yy}(x,a) = 0 = C - \rho \frac{a^2 \Omega^2}{2}$$

<u>Sol'n</u>

$$\sigma_{xx} = \tau_{xy} = 0, \qquad \sigma_{yy} = \rho \frac{a^2 \Omega^2}{2}\left(1 - y^2/a^2\right)$$

(b) Transform

$$\sigma' = A \sigma A^T, \qquad \begin{Bmatrix} e_r \\ e_\theta \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \end{Bmatrix} = A \underset{\sim}{e}$$

$$\sigma_{rr} = \frac{\sigma_{yy}}{2}(1 - \cos 2\theta), \quad \sigma_{\theta\theta} = \frac{\sigma_{yy}}{2}(1 + \cos 2\theta), \quad \sigma_{r\theta} = \frac{\sigma_{yy}}{2}\sin 2\theta$$

$$\sigma_{rr}(r,\theta) = \rho \frac{a^2 \Omega^2}{4}\left(1 - \frac{r^2}{a^2}\sin^2\theta\right)(1 - \cos 2\theta)$$

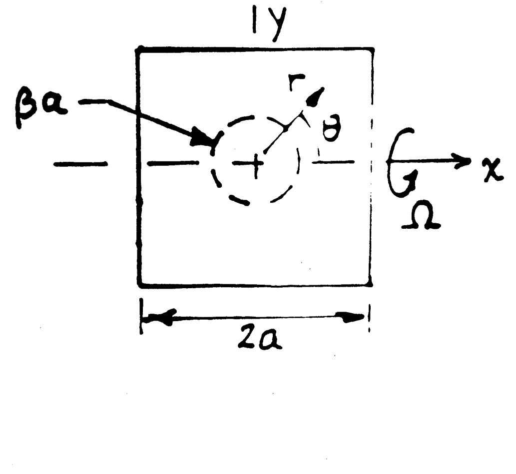
$$\sigma_{r\theta}(r,\theta) = \rho \frac{a^2 \Omega^2}{4}\left(1 - \frac{r^2}{a^2}\sin^2\theta\right)\sin 2\theta$$

$$\sigma_{rr}(\beta a,\theta) = \alpha\left[(4 - 3\beta^2) - 4(1 - \beta^2)\cos 2\theta - \beta^2 \cos 4\theta\right], \qquad \alpha = \rho \frac{a^2 \Omega^2}{16}$$

$$\sigma_{r\theta}(\beta a,\theta) = \alpha\left[(4 - 2\beta^2)\sin 2\theta + \beta^2 \sin 4\theta\right]$$

traction on edge of hole, $r = \beta a$

$$\underset{\sim}{t}(\beta a,\theta) = \sigma_{rr}(\beta a,\theta)\underset{\sim}{e}_r + \sigma_{r\theta}(\beta a,\theta)\underset{\sim}{e}_\theta$$



(c) Airy stress fcn. terms which give these surface tractions and
i. preferably stresses that decrease with r while also
ii matching coefficients of stress components at the edge of hole.

$$\phi = A \ln r + B\cos 2\theta + \frac{C}{r^2}\cos 2\theta + D r^4 \cos 4\theta$$

$$\boxed{\begin{aligned} \sigma_{rr}(r,\theta) &= A r^{-2} - 4B r^{-2}\cos 2\theta - 6C r^{-4}\cos 2\theta - 12 D r^2 \cos 4\theta \\ \sigma_{r\theta}(r,\theta) &= 0 - 2B r^{-2}\sin 2\theta - 6C r^{-4}\sin 2\theta + 12 D r^2 \sin 4\theta \end{aligned}}$$

Equating to traction at edge of hole:

$$\begin{aligned} \sigma_{rr}(\beta a,\theta) &= A(\beta a)^{-2} - 4B(\beta a)^{-2}\cos 2\theta - 6C(\beta a)^{-4}\cos 2\theta - 12D(\beta a)^2\cos 4\theta \\ &= \alpha\left[(4-3\beta^2) - 4(1-\beta^2)\cos 2\theta - \beta^2\cos 4\theta\right] \\ \sigma_{r\theta}(\beta a,\theta) &= 0 - 2B(\beta a)^{-2}\sin 2\theta - 6C(\beta a)^{-4}\sin 2\theta + 12D(\beta a)^2\sin 4\theta \\ &= \alpha\left[(4-2\beta^2)\sin 2\theta + \beta^2\sin 4\theta\right] \end{aligned}$$

Coefficients

$A = \alpha(4-3\beta^2)(\beta a)^2$, $B = -\alpha\beta^2(\beta a)^2$, $C = \frac{2\alpha}{3}(\beta a)^4$

$D = \alpha/12a^2$

(d) not possible to obtain $\sigma_{rr} \sim -\alpha\beta^2\cos 4\theta$ and $\sigma_{r\theta} \sim +\alpha\beta^2\sin 4\theta$ from Pot. Fcn term that decreases with r. Hence, even for $\beta \ll 1$, correction solution affects traction on outer boundary. Solution requires these additional stresses to be eliminated.