Q1 (a) From lecture notes: velocity potential ϕ for the flow satisfies $\nabla^2\phi = 0$

Boundary conditions:

(1) Top surface $\quad \dfrac{\partial^2\phi}{\partial t^2} = -g\dfrac{\partial\phi}{\partial z} \quad$ on $z = 0$

(2) Bottom: deep water, so $u = \nabla\phi \to 0$ as $z \to -\infty$

(3) Side walls: no flow through walls so $\dfrac{\partial\phi}{\partial y} = 0$ on $y = 0, L$

The model neglects viscosity and surface tension. It also assumes that wave amplitudes are small.

(b) Try separable solution $\phi = f(z)g(y)e^{i(\omega t - kx)}$

Then $\nabla^2\phi = 0$ gives $-k^2 fg + f''g + g''f = 0$

$$\therefore \quad \frac{f''}{f} + \frac{g''}{g} - k^2 = 0$$

This is in separated form, so deduce that

$$\frac{g''}{g} = -\mu^2 \text{ say, and } \frac{f''}{f} = k^2 + \mu^2 \quad (\mu \text{ constant})$$

So $g = \begin{matrix}\sin \\ \cos\end{matrix}\Big) \mu y$. But $g'(0) = 0$, so must be cos.

Then $g'(y) = 0$ at $y = L$, so $\sin \mu L = 0$

$$\therefore \mu = \frac{n\pi}{L}, \quad n = 0, 1, 2, 3 \ldots$$

Then $f'' = (k^2 + \mu^2) f$ so $f = e^{\pm\sqrt{k^2+\mu^2}\, z}$

But $f' \to 0$ as $z \to -\infty$ (deep water) so only the + sign is relevant.

So possible solutions are $\phi = \phi_0 e^{\sqrt{k^2+\mu^2}\, z} \cos\mu y \, e^{i(\omega t - kx)}$

with $\mu = \dfrac{n\pi}{L}, \quad n = 0, 1, 2 \ldots$

Now the top surface condition gives

$$-\omega^2 = -g\sqrt{k^2+\nu^2}$$

$$\therefore\ k^2 = \frac{\omega^4}{g^2} - \left(\frac{n\pi}{L}\right)^2 \quad , \quad n = 0, 1, 2 \dots$$

(c) Cut-on frequency where k^2 goes from negative to positive, so where $\dfrac{\omega^2}{g} = \dfrac{n\pi}{L}$.

Phase velocity $c_p = \dfrac{\omega}{k} = \omega \Big/ \sqrt{\dfrac{\omega^4}{g^2} - \left(\dfrac{n\pi}{L}\right)^2}$

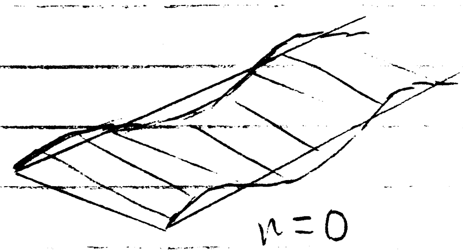
Group velocity $c_g = \dfrac{d\omega}{dk} = \dfrac{d}{dk}\left[g^2(k^2+\nu^2)\right]^{1/4}$

$$= \frac{1}{4}\left[g^2(k^2+\nu^2)\right]^{-3/4} \cdot 2g^2k$$

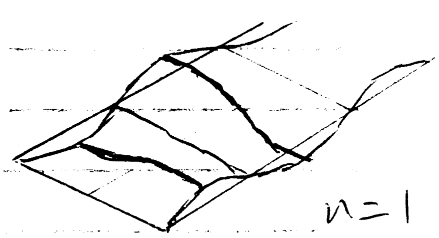
$$= \frac{g^2}{2}\frac{k}{\omega^3} = \frac{g^2}{2\omega^3}\sqrt{\frac{\omega^4}{g^2} - \left(\frac{n\pi}{L}\right)^2}$$

Note that at a cut-on frequency, $c_p \to \infty$, $c_g \to 0$.

(d) $L = 5\,\text{m}$, so cut-on frequencies are
$n=0$, $0\,\text{Hz}$
$n=1$, $0.395\,\text{Hz}$
$n=2$, $0.559\,\text{Hz}$

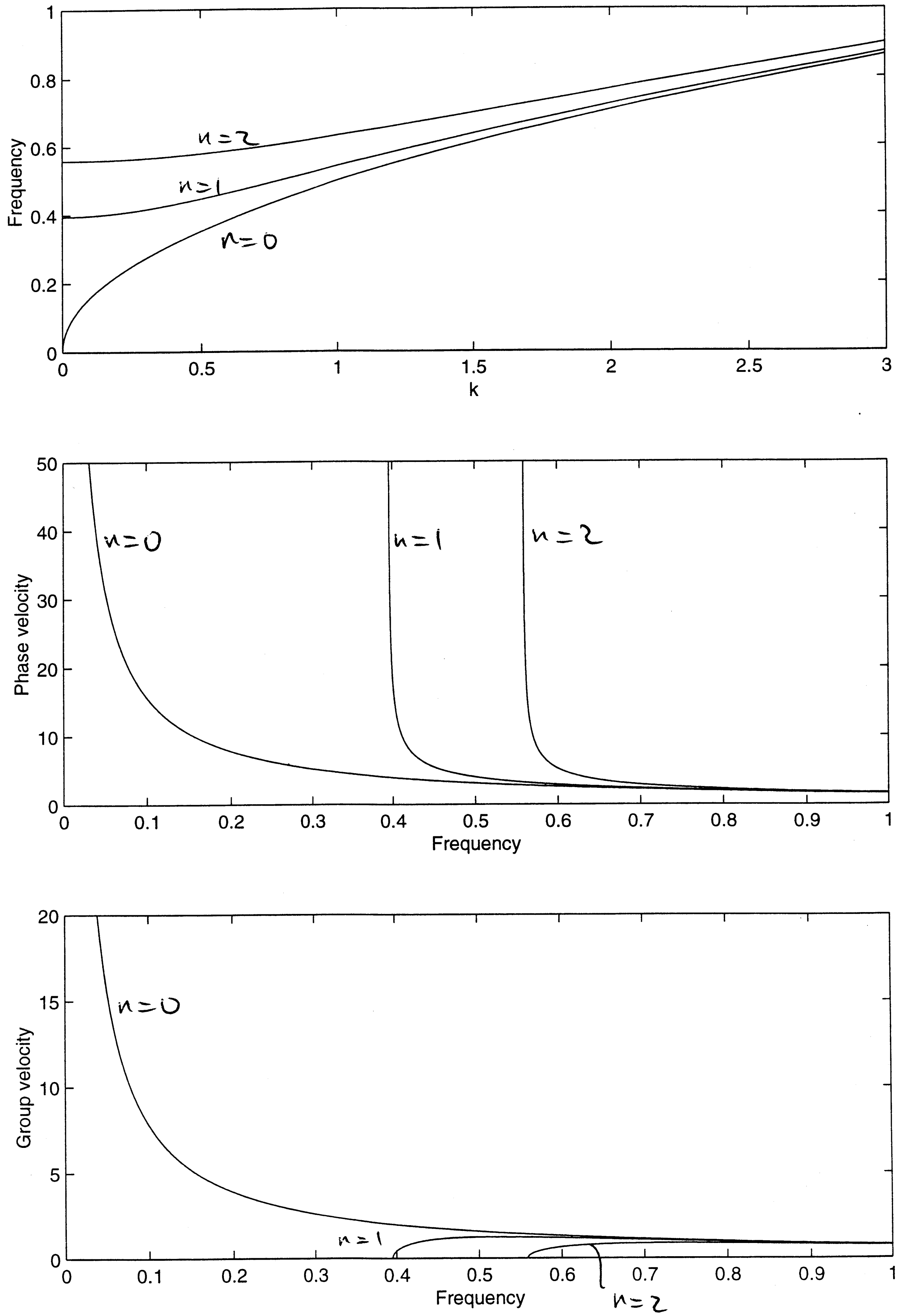


$n=0$



$n=1$

$n=2$

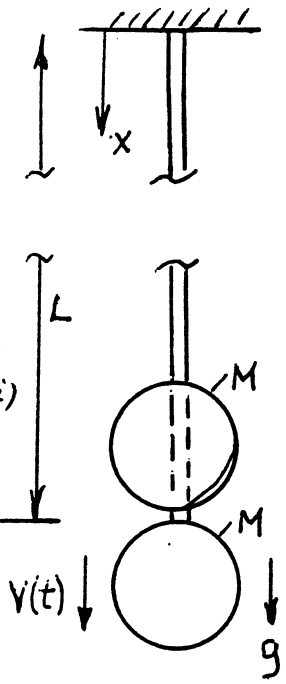


Frequency
k
n=0
n=1
n=2
Phase velocity
Frequency
n=0
n=1
n=2
Group velocity
Frequency
n=0
n=1
n=2

Q2 (a) conservation of translational momentum

$$M\tilde{V}_0 = 2MV_0 \longrightarrow V_0 = \tilde{V}_0/2$$

$$\sigma_0 = \frac{Mg}{A}$$



(b) eq of motion for mass, $t > 0$

$$2M\frac{dV}{dt} = -A\left[\sigma_i(L,t) + \sigma_r(L,t)\right] - A\sigma_0 + 2Mg \quad (i)$$

continuity of velocity

$$V(t) = \dot{u}_i(L,t) + \dot{u}_r(L,t) \quad (ii)$$

cons. momentum across wavefront, $t > 0$

$$\sigma_i = -\rho c_0 \dot{u}_i, \qquad \sigma_r = \rho c_0 \dot{u}_r \quad (iii)$$

$$\therefore \quad \frac{dV}{dt} = \frac{\rho c_0 A}{2M}\left\{\dot{u}_i(L,t) - \left[V(t) - \dot{u}_i(L,t)\right]\right\} + \frac{g}{2}$$

0 for $t < 2L/c_0$

or
$$\frac{dV}{dt} + \frac{c_0}{mL}V = 2\cancel{\dot{u}_i}(L,t) + g/2 \qquad \text{where } m \equiv \frac{2M}{\rho A L}$$

Sol'n
$$V = \frac{gmL}{2c_0} + Be^{-c_0 t/mL}$$

I.C.
$$V_0 = V(0) = \frac{gmL}{2c_0} + B \quad \Rightarrow \quad B = V_0 - \frac{gmL}{2c_0}$$

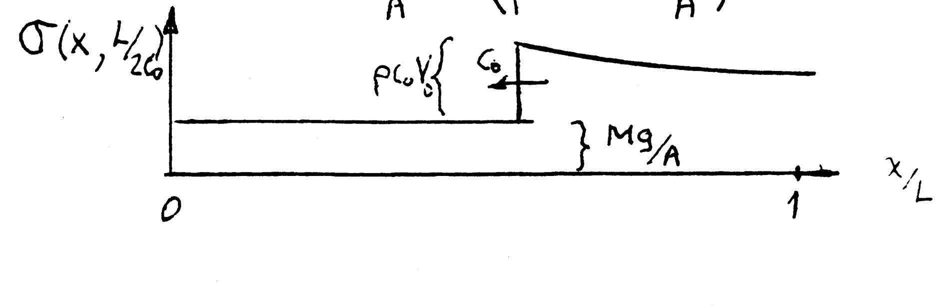
$$\therefore \quad V(t) = \alpha + (V_0 - \alpha)e^{-\beta t}, \qquad \alpha = \frac{gmL}{2c_0}, \quad \beta = \frac{c_0}{mL}$$

$t < 2L/c_0$

(c)
$$\sigma_r(L,t) = \rho c_0 V(t)$$

$$\sigma_r(x,t) = \rho c_0\left\{\alpha + (V_0 - \alpha)e^{-\frac{(c_0 t + x - L)}{mL}}\right\} \qquad x > L - c_0 t$$

$$= \frac{Mg}{A} + \left(\rho c_0 V_0 - \frac{Mg}{A}\right)e^{-\frac{(c_0 t + x - L)}{mL}}$$



(d) at clamped end when wavefront arrives, $t = L/c_0$

$$\sigma(0, L/c_0^{+}) = \sigma_F = \frac{Mg}{A} + 2\rho c_0 V_0 = \frac{Mg}{A} + 2\rho c_0 \frac{\tilde{V}_{0F}}{2}$$

$$\therefore \quad \tilde{V}_{0F} = \frac{\sigma_F}{\rho c_0} - \frac{Mg}{\rho c_0 A} = \frac{\sigma_F}{\rho c_0} - \frac{mgL}{2c_0}$$

break located at $x = 0$ (the clamp) at $t = L/c_0^{+}$

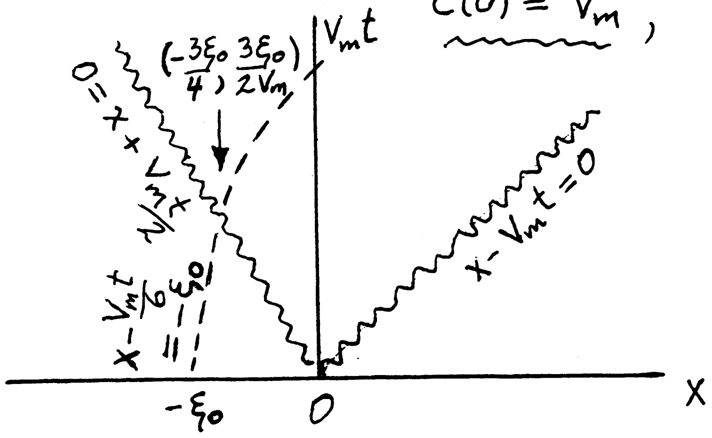
Q3 (a) flux: $q = \rho V = \rho_m V_m \left(y - y^{3/2} \right), \quad y = \rho/\rho_m$

$$0 = \frac{d(q)}{dy} = \rho_m V_m \left(1 - \frac{3}{2} y^{1/2}\right)$$

$\therefore$ $\rho = 4/9\, \rho_m \rightarrow (\text{flux})_{max}$

(b) characteristic speeds $c(\rho) = dq/d\rho = V_m\left(1 - \frac{3}{2}\sqrt{\rho/\rho_m}\right)$

$c(0) = V_m, \quad c(\rho_m) = -V_m/2$



constrained speed

$$V = V_m\left(1 - \sqrt{\rho/\rho_m}\right) = V_m\left(1 - 5/6\right) = V_m/6$$

(c) on characteristic lines in fan emanating from $(0,0)$

$$x = \frac{dq}{d\rho} t = V_m t \left(1 - \frac{3}{2}\sqrt{\rho/\rho_m}\right) \longrightarrow \sqrt{\rho/\rho_m} = \frac{2}{3}\left(1 - \frac{x}{V_m t}\right)$$

speed of bean located at $x = \xi(t)$

$$\frac{d\xi}{dt} = V_m\left[1 - \frac{2}{3}\left(1 - \frac{\xi}{V_m t}\right)\right] = \frac{V_m}{3}\left[1 + \frac{2\xi}{V_m t}\right]$$

$$\therefore 3t \frac{d\xi}{dt} - 2\xi = V_m t$$

sol'n: $\xi = B_1 V_m t + B_2 t^{\eta}$

$d\xi/dt = B_1 V_m + B_2 \eta t^{\eta - 1}$

$3B_1 V_m t + 3B_2 \eta t^{\eta} - 2B_1 V_m t - 2B_2 t^{\eta} = V_m t$

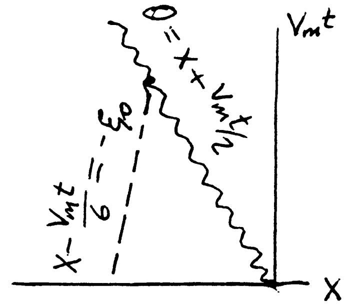
$\therefore$ $B_1 = 1, \quad \eta = 2/3$

$$\xi(t) = V_m t + B_2 t^{2/3}$$

(d) <u>in fan:</u> $\xi(t) = V_m t + B_2 t^{2/3}$

initial cond:

bean enters fan at $\left(-\frac{3\xi_0}{4}, \frac{3\xi_0}{2V_m}\right)$



$$\therefore\ -\frac{3\xi_0}{4} = \frac{3\xi_0 \cancel{V_m}}{2\cancel{V_m}} + B_2\left(\frac{3\xi_0}{2V_m}\right)^{2/3}$$

$$B_2 = -\frac{9\xi_0}{4}\left(\frac{2V_m}{3\xi_0}\right)^{2/3}$$

$$\xi(t) = V_m t - \frac{9\xi_0}{4}\left(\frac{2V_m t}{3\xi_0}\right)^{2/3}, \qquad t > \frac{3\xi_0}{2V_m}.$$