4F1 2004 Solution

1 (a)

$$\sigma \ln|S(\sigma)| = \frac{2}{\pi}\int_0^\infty \frac{\sigma^2}{\sigma^2+\omega^2}\ln|S(j\omega)|\,d\omega \qquad (I)$$

By assumption $L(\sigma) \sim c\sigma^{-k}$ $(k \geq 2)$ for large σ, so

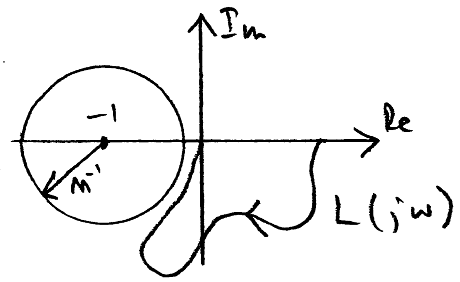
$$\sigma \ln|S(\sigma)| = \sigma \ln(1+L(\sigma))^{-1} \sim -\sigma\ln(1+c\sigma^{-k})$$
$$= -\sigma(c\sigma^{-k}+\dots) = -c\sigma^{-k+1} + \dots$$

Thus $\sigma \ln|S(\sigma)| \to 0$ as $\sigma \to \infty$.
As $\sigma \to \infty$, RHS of (I) tends to $\int_0^\infty \ln|S(j\omega)|\,d\omega$ which completes the proof. (There is a subtle point in the last step, not needed for full marks, that the tail of the integral has a vanishingly small contribution so it doesn't matter that $\sigma^2/(\sigma^2+\omega^2)$ is not close to one if $\omega \gg \sigma$.)

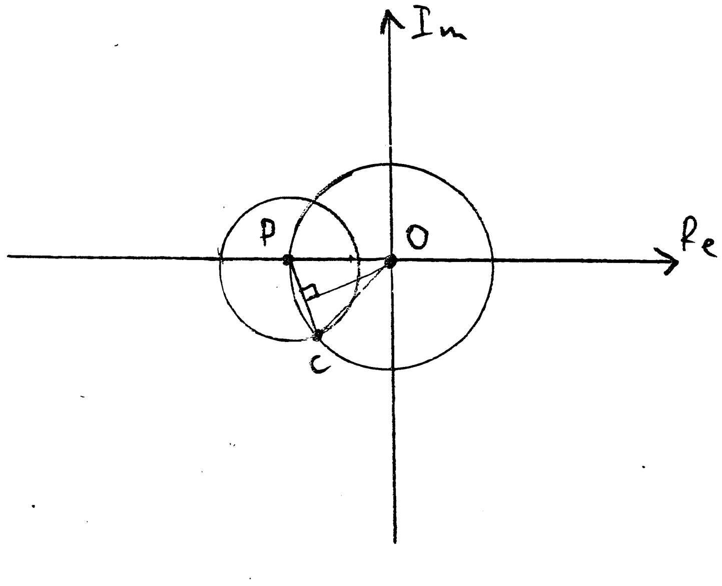
[25%]

(b) $|S(j\omega)| \leq M \iff |1+L(j\omega)| \geq M^{-1}$

$\iff$ $L(j\omega)$ lies outside a circle centre -1 of radius M^{-1}



Consider the intersection of the above circle with the unit circle centre the origin.



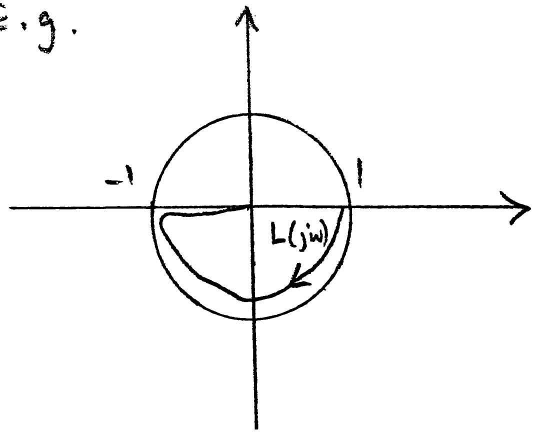
Phase margin $= \alpha = \angle POC$

$PC = M^{-1}$, $PO = PC = 1$

$\Rightarrow \sin\left(\frac{\alpha}{2}\right) = \frac{M^{-1}}{2}$

$\Rightarrow \alpha = 2\sin^{-1}\left(\frac{1}{2M}\right)$

A lower bound on Phase Margin does not guarantee an upper bound on M. E.g.



This $L(j\omega)$ has an infinite PM but poor sensitivity.

[25%]

(c) For $\omega \geq 10$, $|L(j\omega)| \leq \left|\frac{1}{(j\omega+1)^2}\right| = \frac{1}{1+\omega^2}$

$$\Rightarrow \quad |S(j\omega)| \leq \frac{1}{1-|L(j\omega)|} \leq \frac{1}{1-(1+\omega^2)^{-1}}$$

Hence

$$\int_{10}^{\infty} \ln|S(j\omega)|\,d\omega \leq -\int_{10}^{\infty} \ln\left(1-(1+\omega^2)^{-1}\right) d\omega$$

$$= -\left[\omega \ln\left(1-(1+\omega^2)^{-1}\right) - 2\tan^{-1}\omega\right]_{10}^{\infty}$$

$$= \pi + 10\ln\left(1-(1.01)^{-1}\right) - 2\tan^{-1}10$$

$$= 0.09983 < 0.0999 < 0.1$$ [25%]

[Note that $\omega \ln(1-(1+\omega^2)^{-1}) \sim -\omega(1+\omega^2)^{-1} + \ldots$ which tends to zero as $\omega \to \infty$.]

(d) Thus

$$0 = \int_0^1 \ln|S(j\omega)|\,d\omega + \int_1^{10} \ln|S(j\omega)|\,d\omega + \int_{10}^{\infty} \ln|S(j\omega)|\,d\omega$$

$$\leq \ln 0.1 + 9\ln M + 0.1$$

$$\Rightarrow \quad M \geq 1.2773 = M_0$$ [25%]

we don't know that this M_0 is achievable so we can't say anything about the achievable phase margin. However, if M_0 was achievable then a phase margin of 46° would be guaranteed using Part (b).

2 (a) (i) $Y(s) = R(s)\frac{1}{s}$

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

$Y(1) = R(1) = 0$. Hence

$$0 = \int_0^\infty y(t)\, e^{-t}\, dt.$$

The final value of the step response is one since $R(0) = 1$. The integral implies $y(t)$ must be negative at some times, though not necessarily immediately after $t > 0$ (which is usually termed undershoot). [20%]

(ii) $\mathcal{L}(\dot{y}(t)) = sY(s)$

Initial Value Theorem:

$$\lim_{t\to 0} \dot{y}(t) = \lim_{s\to\infty} s\,(sY(s)) = \lim_{s\to\infty} s\,R(s)$$

For this to be zero we require $m \geqslant 2$ since

$$\lim_{s\to\infty} s\,\frac{N(s)}{D(s)} \sim \lim_{s\to\infty} s\, s^{-m}$$

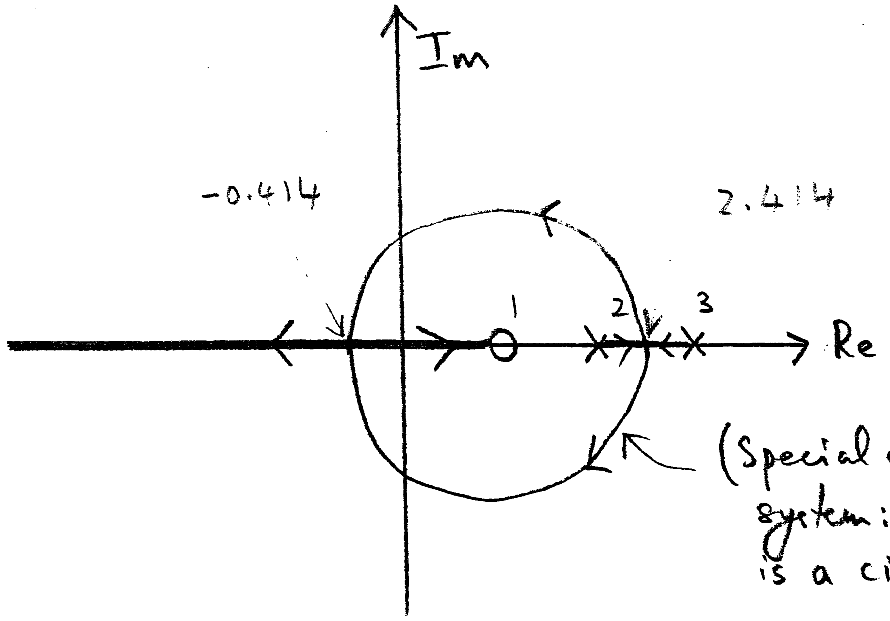
[15%]

(b) (i) $\dfrac{s-1}{s^2-5s+6} = \dfrac{s-1}{(s-2)(s-3)}$

Breakaway pts: $s^2-5s+6 - (s-1)(2s-5) = 0$

$\iff s^2-5s+6-(2s^2-7s+5) = 0$

$\iff s^2-2s-1 = 0 \iff s = 1 \pm \sqrt{2}$



(Special case of 2^{nd} order system: off-axis portion is a circle)

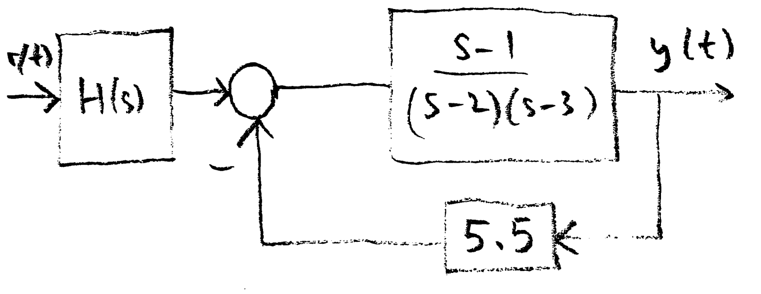
Closed-loop poles: $\quad s^2 - 5s + 6 + k(s-1)$
$$\equiv s^2 + (k-5)s + 6 - k$$

Closed-loop system will be stable providing $5 < k < 6$ [30%]

(ii) To achieve critical damping we choose k so that closed-loop poles are given by:
$$(s + \sqrt{2} - 1)^2 = s^2 + (2\sqrt{2} - 2)s + 3 - 2\sqrt{2}$$

$$\Rightarrow k = 3 + 2\sqrt{2} = 5.828$$

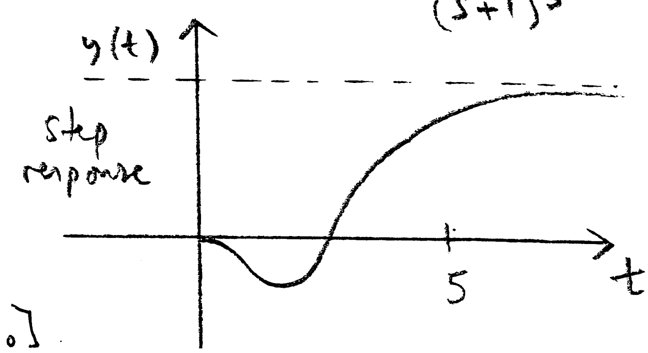
Or choose k in middle of range for largest gain margin. [25%]



$$R(s) = H(s)\left(\frac{s-1}{s^2 + 0.5s + 0.5}\right)$$

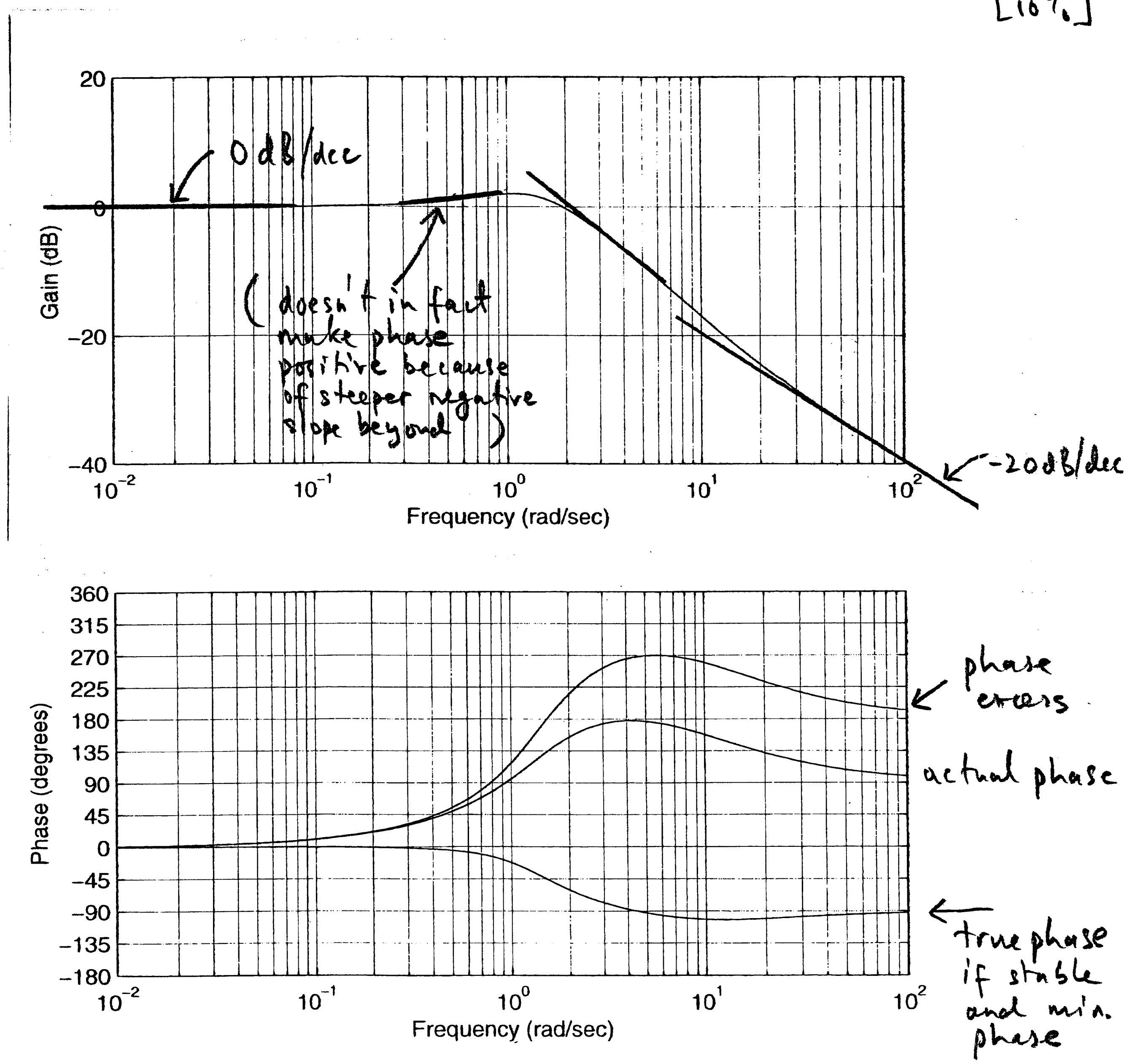
$$R(s) = \frac{1-s}{(s+1)^3} \text{ meets all specs} \Rightarrow H(s) = \frac{-(s^2 + 0.5s + 0.5)}{(s+1)^3}$$

(iii) Zero initial slope, undershoot and final value of one are expected. Absence of overshoot and scaling of time axis are not required. [10%]



3 (a)(i) General form of min. phase (starting at zero and tending to -90° for large w) and phase excess (starting at zero, exceeding +180° in mid-range and tending to +180° for large w) is sufficient for full credit.

[10%]



(ii) If there are near pole-zero cancellations in the right half plane they would be scarcely detectable in the phase plot, so we can only determine a least number of RHP poles (and zeros).

[20%]

At least two RHP poles are required for phase excess to exceed 180°. Plot is consistent with 2 RHP poles and 1 RHP zero. Estimate poles ~ 1 rad/s and zero ~ 10 rad/s.

[Actual plant: $2(s+1)(-0.08s+1)/(s^2-2s+2)(0.15s+1)$]

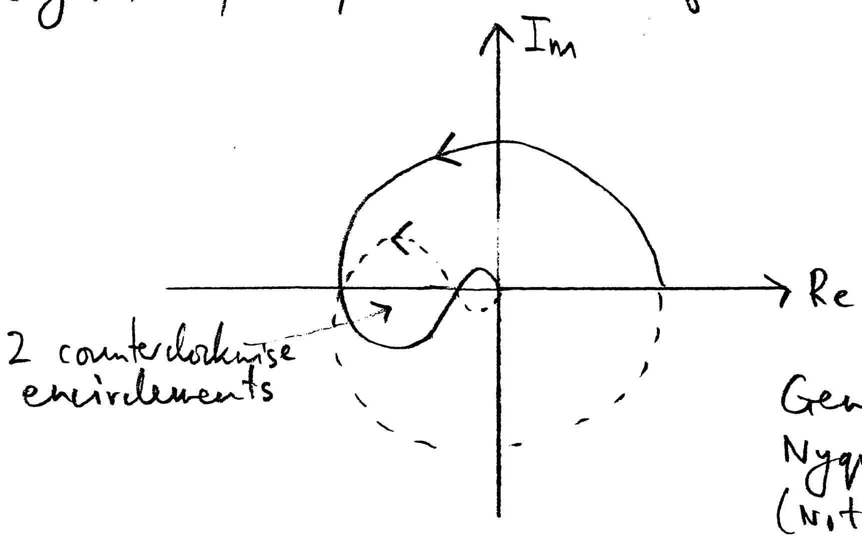
(iii) Expect the achievable crossover frequency to lie between 1 rad/sec and 10 rad/sec. [10%]

(b) A phase lead compensator is needed to increase phase beyond +180° to give 2 counter-clockwise encirclements. Try:

$$K(s) = K_1(s) = 2\alpha \frac{(s + w_c/\alpha)}{s + w_c \alpha} \qquad w_c = 4, \quad \alpha = 3$$

$$= 6 \frac{(s + 4/3)}{s + 12}$$

which gives a peak phase advance of 53° at 4 rad/sec.



General form of Nyquist diagram. (Not to scale.)

New crossover is in fact just below 4 rad/sec [30%]

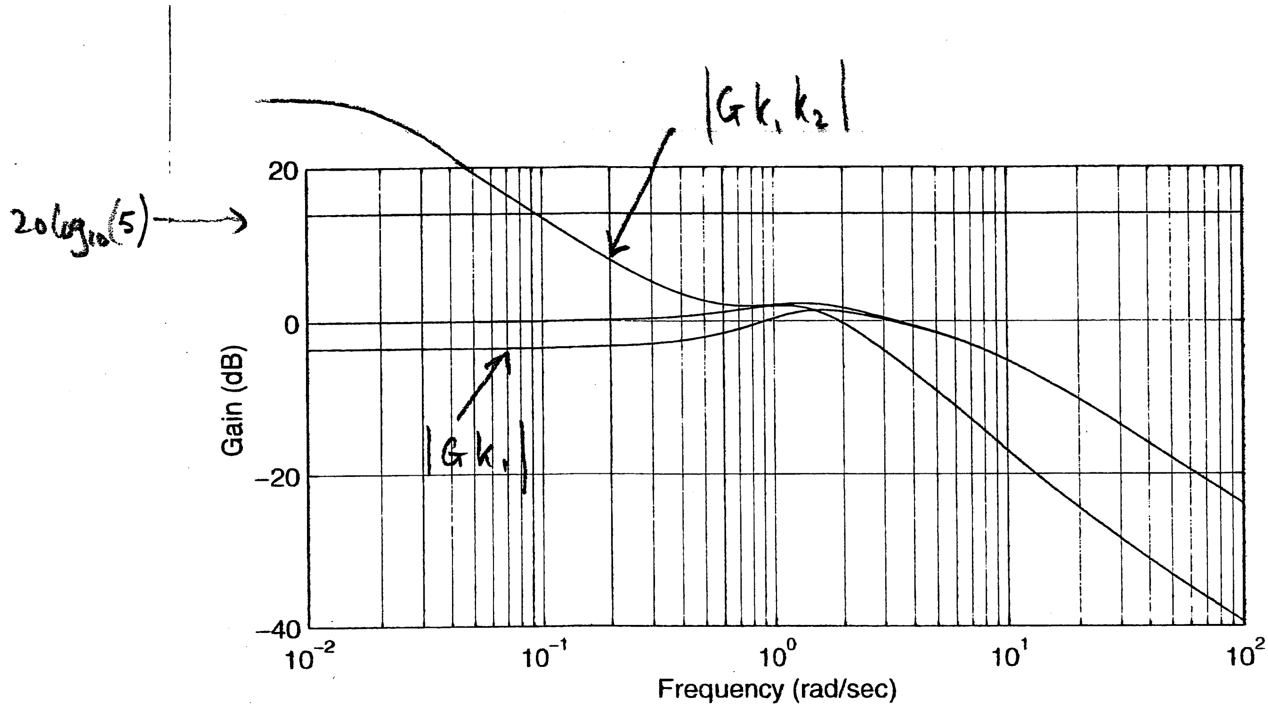
(c) This spec can be achieved with a lag compensator,

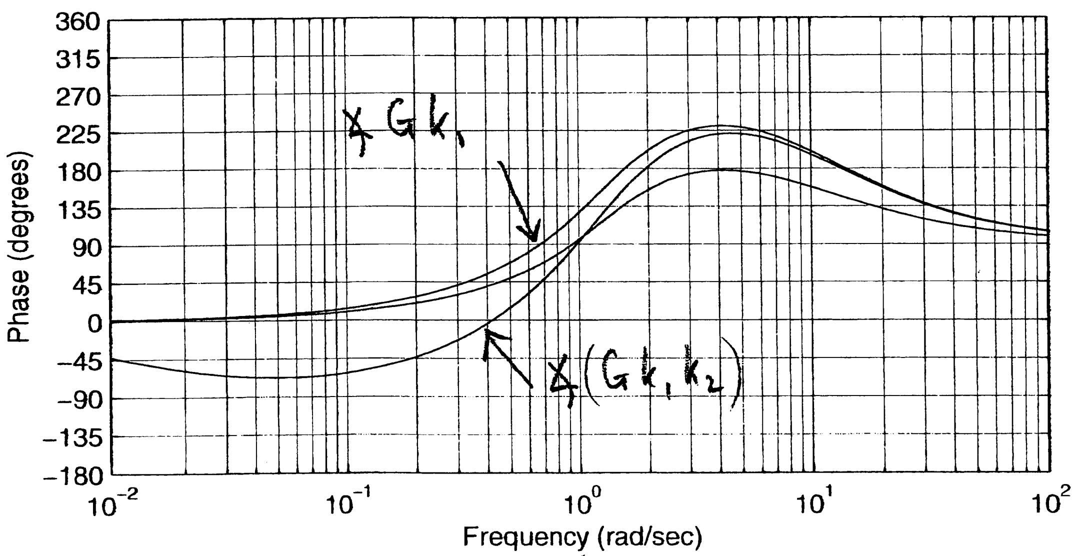
$$K_2(s) = \frac{s + 0.7}{s + 0.01}$$

is sufficient to raise gain at $w = 0.1$ by about a factor of 10. Zero at 0.7 is as low as possible to avoid too much phase lag around crossover.

$$\angle K_2(j4) = -9.8°$$

So new phase margin will be around 40°.





Final compensator: $k(s) = k_1(s)k_2(s)$

[30%]