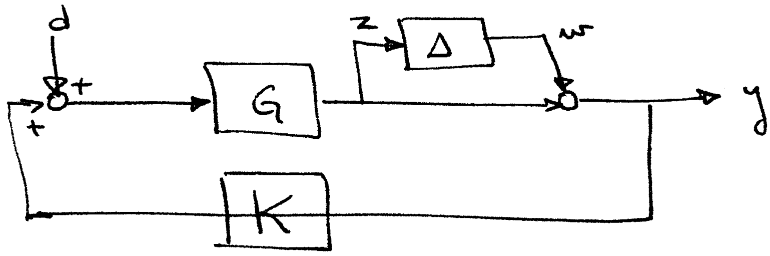
1)

4F2 - 2004

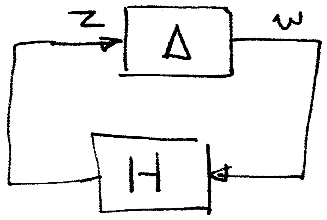


(a) $z = G(d + K(w+z))$

$\Rightarrow (I - GK)z = Gd + GKw$

$\Rightarrow z = \underbrace{(I-GK)^{-1}GK}_{=H} w + (I-GK)^{-1}G d.$

To assess robust stability let $d=0$ when



and H will be stable by assumption of stability when $\Delta = 0$. The small gain Theorem states that this feedback system will be internally stable for all stable Δ, with $\|\Delta\|_\infty < \varepsilon$ if <u>and only if</u> $\|H\|_\infty \leq 1/\varepsilon$.

(b) $H(0) = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

(i) $\bar{\sigma}^2(H(0)) = \lambda_{max}(H(0)H(0)^*)$

Let $L = H(0)H(0)^* = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix}$

For eigenvalues of L solve,

$0 = \det(\lambda I - L) = \det\begin{bmatrix} \lambda-13 & -6 \\ -6 & \lambda-4 \end{bmatrix} = (\lambda-13)(\lambda-4) - 36$

$= \lambda^2 - 17\lambda + 16 = (\lambda-1)(\lambda-16) \Rightarrow \lambda = 16$ OR 1.

Hence the singular values of $H(0)$ are 4 and 1

$\Rightarrow \bar{\sigma}(H(0)) = 4.$

(ii) By part (a) the closed-loop will be stable for all $\|\Delta\|_\infty < \epsilon$ if and only if $\|H\|_\infty \leq 1/\epsilon$ so that if $\|H\|_\infty = 4$ there will exist Δ with $\|\Delta\|_\infty = 1/4$ that will be destabilizing. Let

$$\Delta = \frac{1}{20}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\sqrt{5}} \frac{\begin{bmatrix} 2 & 1 \end{bmatrix}}{\sqrt{5}} \cdot \frac{1}{4}$$

$\Rightarrow \|\Delta\|_\infty = 1/4$

Also $\det(I - H(0)\Delta(0))$

$$= \det\left(I - \begin{pmatrix} 2 & 3 \\ 6 & 2 \end{pmatrix} \frac{1}{20}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\right)$$

$$= \det\left(I - \frac{1}{20}\begin{bmatrix} 16 & 8 \\ 8 & 4 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix} = 0$$

hence $I - H(s)\Delta(s)$ is not invertible at $s = 0$

so there is a closed loop pole at $s = 0$

and the perturbed system is not stable.

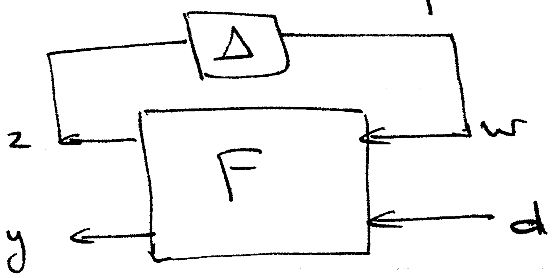
(c)

$$z = (I-GK)^{-1}GK\,w + (I-GK)^{-1}G\,d$$

$$\begin{aligned} y &= z + w \\ &= \left[I + (I-GK)^{-1}GK\right] w + (I-GK)^{-1}G\,d \\ &= (I-GK)^{-1}\{I - GK + GK\}\, w + (I-GK)^{-1}G\,d \\ &= (I-GK)^{-1} w + (I-GK)^{-1}G\,d \end{aligned}$$

~~Let~~

$$\Rightarrow \begin{bmatrix} z \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} (I-GK)^{-1}GK & (I-GK)^{-1}G \\ (I-GK)^{-1} & (I-GK)^{-1}G \end{bmatrix}}_{F} \begin{bmatrix} w \\ d \end{bmatrix}$$



The robust performance test is hence

$$\cancel{\mu(F)}\ \mu\left\{\begin{bmatrix} \epsilon & \\ & 1/M \end{bmatrix} F\right\} \le 1$$

~~where~~ with respect to the structure $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$.

2) a) $V(x,k) = \min_{u_k,\dots} \left(\sum_k c(x_k, u_k) + J_h(x_h) \right)$.

$V(x,0)$ is the optimal cost

$V(x,h)$ is the terminal cost

Bookwork, or "If we know the optimal cost from the $k+1$th stage, the optimal cost from now is clearly the minimum over the current input of the sum of the incremental cost associated with that input and the optimal cost from where we end up."

Reduces single minimization over h variables to h 1-dimensional minimizations.

b) i) $V(x,k) = \min_u \left(u^2 + V(x+u, k+1) \right)$

ii) $V(x+u, k+1) = f(k+1)(x+u)^2$

$$\Rightarrow \min_u \; u^2 + \cancel{x^2 f(k+1)} \; (x^2 + u^2 + 2xu) f(k+1)$$

$$= \min_u \left(\sqrt{1+f(k+1)}\, u + \frac{x f(k+1)}{\sqrt{1+f(k+1)}} \right)^2 + \left(f(k+1) - \frac{f^2(k+1)}{1+f(k+1)} \right) x^2$$

$$= \frac{f(k+1)}{1+f(k+1)} x^2$$

$$\Rightarrow f(k) = \frac{f(k+1)}{1+f(k+1)}$$

iii) $V(x,5) = x_5 \Rightarrow f(5) = 1$

$\Rightarrow f(4) = \frac{1}{2}$

$\Rightarrow f(3) = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$

$\Rightarrow f(2) = \frac{1}{4}$

$\Rightarrow f(1) = \frac{1}{5}$

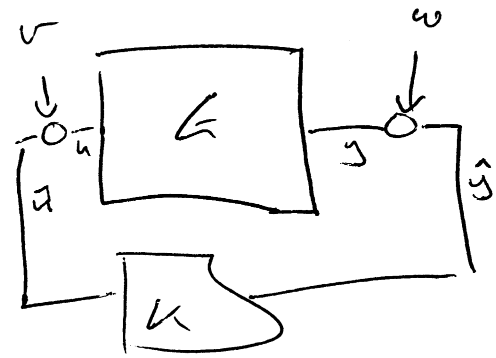
$\Rightarrow f(0) = \frac{1}{6}$ etc

$u_k = -\frac{x_k f(k+1)}{1+f(k+1)}$

$u_k = - x_k f_k$

$\Rightarrow u_k = -\frac{1}{6} \quad \forall k$

3) a)



$$y = G(v+\hat{u})$$
$$u = v + \hat{u}$$
$$\hat{y} = G(v+\hat{u}) + w$$

$$\Rightarrow \quad P(s): \quad \begin{bmatrix} y \\ u \\ \hat{y} \end{bmatrix} = \begin{bmatrix} 0 & G & G \\ 0 & I & I \\ I & G & G \end{bmatrix} \begin{bmatrix} w \\ v \\ \hat{u} \end{bmatrix}$$

$$F_l(P,K) = \begin{bmatrix} 0 & G \\ 0 & I \end{bmatrix} + \begin{bmatrix} G \\ I \end{bmatrix} K(I-KG)^{-1}\begin{bmatrix} I & G \end{bmatrix}$$

$$= \begin{bmatrix} GK(I-KG)^{-1} & G + GK(I-KG)^{-1}G \\ K(I-KG)^{-1} & I + K(I-KG)^{-1}G \end{bmatrix}$$

$$= \begin{bmatrix} G(I-GK)^{-1}K & G(I-GK)^{-1} \\ (I-GK)^{-1}K & G(I-GK)^{-1} \end{bmatrix}$$

b)

$$\begin{bmatrix} \dot{x} \\ y \\ u \\ \hat{y} \end{bmatrix} = \begin{array}{c} \left[\begin{array}{c|ccc} a & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} x \\ w \\ v \\ \hat{u} \end{array}$$

c) CARE

$$2aX + 1 - Xk = 0$$

$$\Rightarrow \quad X^2k^2 - 2aX - 1 = 0$$

$$\Rightarrow \quad X = \frac{2a \pm \sqrt{4a^2 + 4k^2}}{2k^2} = \frac{a \pm \sqrt{a^2+k^2}}{k^2}$$

Need "$A - BB^TX$" stable

ie $\quad a - \cancel{k^2}\left(\frac{a \pm \sqrt{a^2+k^2}}{\cancel{k^2}}\right) < 0$

$$\Rightarrow \quad X = \frac{a + \sqrt{a^2+k^2}}{k^2}$$

FARE $\quad Y^2 - 2aY - k^2 = 0$

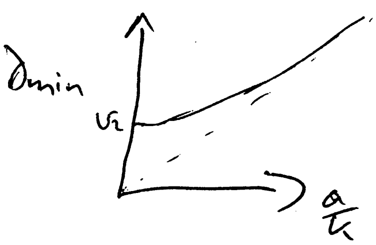
$$\Rightarrow \quad Y = \frac{2a \pm \sqrt{4a^2+4k^2}}{2}$$

$$= a \pm \sqrt{a^2+k^2}$$

$Y = a + \sqrt{a^2+k^2}$ is stabilizing sol^n, as above

$$\gamma_{min} = \sqrt{1 + \bar{\lambda}(XY)} = \sqrt{1 + \frac{\left(a + \sqrt{a^2+k^2}\right)^2}{k^2}}$$

$$= \sqrt{1 + \left(\frac{a}{k} + \sqrt{\left(\frac{a}{k}\right)^2 + 1}\right)^2}$$



$\Rightarrow$ To maintain reasonable γ_{min}, need k to increase with a

d) gap or ν-gap ball G_D : $\delta_\nu(G, G_D) < \frac{1}{\gamma_{min}}$
(or normalized coprime factors)