A.1

(a) Bookwork — Indirect method; Direct method — stability, — asymptotic stability, [— Lasalle's theorem]

(b) (i) $\dot{x}_1 = 0 \Rightarrow x_1 = x_2$

$$\therefore \dot{x}_2 = \cancel{x_2} - x_1\left[1+(\alpha+\beta)^2 x_1^2\right]$$

$\therefore \dot{x}_2 = 0 \Rightarrow x_1 = 0$ or $x_1^2 = -\frac{1}{(\alpha+\beta)^2}$ which has no real solution.

Hence $(x_1 = 0,\ x_2 = 0)$ is the only equilibrium.

(ii) $V(x) = x_1^2 + x_2^2 \Rightarrow \nabla V = [2x_1, 2x_2]^T$

$$\therefore \dot{V} = \nabla V \cdot \underline{f}(\underline{x}) = [2x_1, 2x_2]\begin{bmatrix} x_2 - x_1 \\ -x_1 - (\alpha x_1 + \beta x_2)^2 x_2 \end{bmatrix}$$

$$= \left[2\cancel{x_1 x_2} - 2x_1^2\right] + \left[-\cancel{2x_2 x_1} - 2x_2^2(\alpha x_1 + \beta x_2)^2\right]$$

$$= -2\left(x_1^2 + x_2^2[\alpha x_1 + \beta x_2]^2\right) \leq 0$$

In fact $x_1 \neq 0$ or $x_2 \neq 0 \Rightarrow \dot{V} < 0$ (if $\beta \neq 0$).

$\therefore$ asymptotically stable.

But $V > 0$, $\dot{V} < 0$ is true for any $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, hence globally asymptotically stable.

(iii) If $\beta = 0$ then

$$\dot{V} = -2x_1^2\left(1 + x_2^2 \alpha^2\right)$$

Hence $x_1 = 0 \Rightarrow \dot{V} = 0$

So Lyapunov's theorem proves stability, but not a.s.

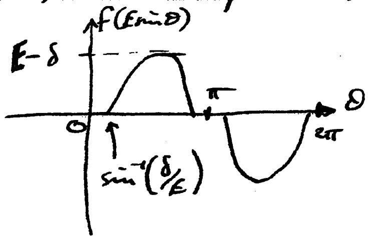
But suppose $x_1^{(t)} = 0$, $x_2^{(t)} \neq 0$. Then $\dot{x}_1 \neq x_2$, so $\dot{x}_1(t) \neq 0$

$\therefore \exists \tau > 0$, st $x_1(t+\tau) \neq 0 \Rightarrow \dot{V}(x(t+\tau)) < 0$.

$\therefore$ by Lasalle's Theorem, it is asy. stable.

(c) Bookwork. Examples: Domestic heating system, Air traffic control

A-2 (a) $N(E) = \dfrac{u_1 + j v_1}{E}$. $\quad v_1 = 0$ since $f(e)$ is an odd function.



Assume $E > \delta$

$$f(E\sin\theta) = \begin{cases} 0 & \text{if } 0 < E\sin\theta < \delta \\ E\sin\theta - \delta & \text{if } \delta < E\sin\theta < \pi/2 \end{cases}$$

$$\therefore u_1 = \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)\sin\theta\, d\theta$$

$$= \frac{4}{\pi}\int_0^{\pi/2} f(E\sin\theta)\sin\theta\, d\theta = \frac{4}{\pi}\int_{\sin^{-1}(\delta/E)}^{\pi/2} (E\sin\theta - \delta)\sin\theta\, d\theta$$

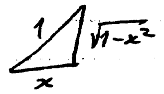
$$= \frac{4}{\pi}\int_{\sin^{-1}(\delta/E)}^{\pi/2} \left[E\sin^2\theta - \delta\sin\theta\right] d\theta$$

$$= \frac{2E}{\pi}\int_{\sin^{-1}(\delta/E)}^{\pi/2} [1 - \cos 2\theta]\, d\theta - \frac{4\delta}{\pi}\int_{\sin^{-1}(\delta/E)}^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{2E}{\pi}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\sin^{-1}(\delta/E)}^{\pi/2} + \frac{4\delta}{\pi}\left[\cos\theta\right]_{\sin^{-1}(\delta/E)}^{\pi/2}$$

$$= \frac{2E}{\pi}\left[\frac{\pi}{2} - 0 - \sin^{-1}(\delta/E) + \tfrac{1}{2}\sin\left[2\sin^{-1}(\delta/E)\right]\right] + \frac{4\delta}{\pi}\left[-\cos\left[\sin^{-1}(\delta/E)\right]\right]$$

But $\cos(\sin^{-1}x) = \sqrt{1-x^2}$



and $\sin(2\sin^{-1}x) = 2\sin(\sin^{-1}x)\cos(\sin^{-1}x) = 2x\sqrt{1-x^2}$

$$\therefore u_1 = \frac{2E}{\pi}\left[\frac{\pi}{2} - \sin^{-1}(\delta/E) + \left(\frac{\delta}{E}\right)\sqrt{1-(\delta/E)^2}\right] - \frac{4\delta}{\pi}\sqrt{1-(\delta/E)^2}$$

$$\therefore N(E) = \frac{u_1}{E} = \frac{2}{\pi}\left[\frac{\pi}{2} - \sin^{-1}(\delta/E) + \left\{\frac{\delta}{E} - \frac{2\delta}{E}\right\}\sqrt{1-(\delta/E)^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - \sin^{-1}(\delta/E) - \frac{\delta}{E}\sqrt{1-(\delta/E)^2}\right]$$

A.2 contd.

(b) If $E < \delta$ then $f(E\sin\theta) \equiv 0$. Hence $\underline{\underline{N(E) = 0}}$.

(c) Suppose $E \to \infty$. Then $N(E) \to \frac{2}{\pi}\left[\frac{\pi}{2} - 0 - 0\right] = \underline{\underline{1}}$.

Intuitive explanation: As $E \to \infty$ then $f(E\sin\theta) = E\sin\theta - \delta$ (for 'almost all' θ)

and $\frac{E\sin\theta - \delta}{E\sin\theta} \to 1$, so $\frac{\text{output}}{\text{input}} \to 1$

(Dead-zone becomes insignificant for very large input signals.)

(d) $0 \leq N(E) < 1$, and $N(E)$ is real.

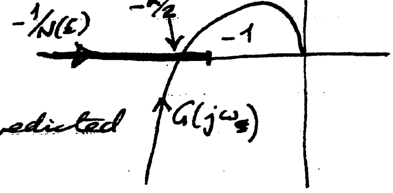
Hence $-\infty < -\frac{1}{N(E)} < -1$, and $-\frac{1}{N(E)} \to -1$ as $E \to \infty$ (from (c)).

So a limit cycle oscillation will be predicted if the Nyquist locus of $G(s)$ intersects the real axis to the left of -1.

Now $G(j\omega) = \frac{k}{j\omega(j\omega+1)^2}$, hence $\angle G(j\omega) = -\frac{\pi}{2} - 2\tan^{-1}(\omega)$

Suppose $\angle G(j\omega_0) = -\pi$, then $-\frac{\pi}{2} - 2\tan^{-1}(\omega_0) = -\pi$
so $\tan^{-1}(\omega_0) = \frac{\pi}{4}$, hence $\omega_0 = 1$.

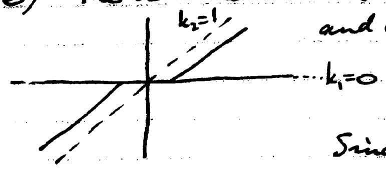
Now $|G(j\omega_0)| = \frac{k}{1 \times (\sqrt{2})^2} = \frac{k}{2}$.



So a limit cycle oscillation is predicted if $\underline{\underline{k > 2}}$

Because of the sense in which the two loci intersect, and $G(s)$ being stable, the limit cycle is predicted to be $\underline{\underline{\text{unstable}}}$.

(e) The dead-zone is a 'sector nonlinearity' in the sector $[0, 1]$ and $G(s)$ is stable.



$\therefore$ circle criterion: $G(j\omega)$ must not penetrate the 'circle' $Re(s) \leq -1$.
Since $Re\{G(j\omega)\} > -2k$, this is assured if $\underline{k < \frac{1}{2}}$.

Popov: Popov locus of $G(s)$ must lie to the right of a line through -1 with +ve slope. $Re\{G(j\omega)\}$ is increasing, so is $\omega\, Im\{G(j\omega)\}$. Hence satisfied if $G^*(j\omega)$ crosses real axis to the right of -1. Hence $\underline{k < 2}$. (Take this as a conservative limit.)

SOLUTION Question 3

3 (a)

$$x_1 = 3x_0 + u_0$$
$$x_2 = 3x_1 + u_1$$
$$\Rightarrow \quad x_2 = 3(3x_0 + u_0) + u_1$$
$$\Rightarrow \quad x_2 = 9x_0 + 3u_0 + u_1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} x_0 + \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$x_0 = x(k) \Rightarrow \Phi = \begin{bmatrix} 3 \\ 9 \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

3 (b) $V(x(k), u) = P x_2^2 + \sum_{s=0}^{1} x_s^2 + u_s^2$

$$= P x_2^2 + x_0^2 + x_1^2 + u_0^2 + u_1^2$$

$(x_0 = x(k))$

$(x_0 = x(k))$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + x_0^2$$

$$+ \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$= u^T u + X^T \Omega X + x(k)^2$$

$$= \cancel{u^T u + (\Phi x(k))^T \Omega X + x(k)^2}$$

$$= u^T u + (\Phi x(k) + \Gamma u)^T \Omega (\Phi x(k) + \Gamma u) + x(k)^2$$

$$= u^T u + x(k)^2 + (u^T \Gamma^T + x(k)^T \Phi^T)(\Omega \Phi x(k) + \Omega \Gamma u)$$

3b) cont...

$$V(x(k),u) = u^T u + x(k)^2 + u^T \Gamma^T \Omega \Phi x(k)$$
$$+ u^T \Gamma^T \Omega \Gamma u + x(k)^T \Phi^T \Omega \Phi x(k)$$
$$+ x(k)^T \Phi^T \Omega \Gamma u$$
$$= u^T (I + \Gamma^T \Omega \Gamma) u + 2u^T \Gamma^T \Omega \Phi x(k)$$
$$+ x(k)^2 + x(k)^T \Phi^T \Omega \Phi x(k)$$

(but $x(k)$ is scalar)

$$\Rightarrow \quad V(x(k),u) = u^T (I + \Gamma^T \Omega \Gamma) u + 2u^T \Gamma^T \Omega \Phi x(k)$$
$$+ (1 + \Phi^T \Omega \Phi) x(k)^2$$

3(c) Need to set $\nabla_u V(x(k),u) = 0$

and solve for u.

$$\nabla_u V(x(k),u) = 2(I + \Gamma^T \Omega \Gamma) u + 2 \Gamma^T \Omega \Phi x(k)$$

$$\Rightarrow (I + \Gamma^T \Omega \Gamma) u = - \Gamma^T \Omega \Phi x(k)$$

But $\Omega = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ and $P > 0$

$$\Rightarrow \Omega > 0 \Rightarrow I + \Gamma^T \Omega \Gamma > 0$$

$$\Rightarrow (I + \Gamma^T \Omega \Gamma)^{-1} \text{ exists}$$

$$\Rightarrow \quad u^*(x(k)) = -(I + \Gamma^T \Omega \Gamma)^{-1} \Gamma^T \Omega \Phi x(k)$$

Check: $\nabla_u^2 V(x(k),u) = 2(I + \Gamma^T \Omega \Gamma) > 0$

$\Rightarrow u^*(x(k))$ is a minimiser

3d) An expression for the RHC law is

$$u(k) = -[1 \quad 0](I + \Gamma^T \Omega \Gamma)^{-1} \Gamma^T \Omega \Phi x(k)$$

3e) The following conditions are sufficient for the RHC law to be stabilising:

a) The terminal cost $F(\cdot)$ is continuous and $F(x) > 0 \ \forall x \neq 0$
b) The stage cost $\ell(\cdot)$ is continuous an $\ell(x,u) > 0 \ \forall (x,u) \neq 0$
c) The terminal control law $u = Kx$ is such that $A+BK$ is stable
d) The terminal cost is a control Lyapunov function in the sense that

$$\forall x : \quad F((A+BK)x) - F(x) \leq -\ell(x, Kx)$$

3 (f) For the given system with $A = 3$ and $B = 1$:

a) $F(x) = Px^2 > 0 \quad \forall x \neq 0$ because $P > 0$
b) $\ell(x,u) = x^2 + u^2 > 0 \quad \forall (x,u) \neq 0$.
c) System has one state $\Rightarrow$
$\rho(A+BK) < 1 \Leftrightarrow |A+BK| < 1 \Leftrightarrow |3+K| < 1$
$\Leftrightarrow -1 \leq 3 + K \leq 1$
$\Leftrightarrow -4 \leq K \leq -2$
d) $P(A+BK)^2 x^2 - Px^2 \leq -x^2 - K^2 x^2 \quad \forall x$

$\Leftrightarrow P(A+BK)^2 - P \leq -1 - K^2$ because $x^2 \geq 0$
$\Leftrightarrow P(3+K)^2 - P \leq -1 - K^2$
$\Leftrightarrow P(K^2 + 6K + 9) - P \leq -1 - K^2$
$\Leftrightarrow P(K^2 + 6K + 8) \leq -1 - K^2$.

Question 4

(a)

$$x_1 = 2x_0 + u_0$$
$$x_2 = 2x_1 + u_1$$
$$\Rightarrow \quad x_2 = 2(2x_0 + u_0) + u_1$$
$$\Rightarrow \quad x_2 = 4x_0 + 2u_0 + u_1$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_0 + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$x_0 = x(k) \Rightarrow \underline{\Phi} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

(b) Constraints on state:

$$x_1 \leq 2$$
$$-x_1 \leq 2$$
$$x_2 \leq 0.8$$
$$-x_2 \leq +1.6$$

$$\Leftrightarrow \quad \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 2 \\ +0.8 \\ +1.6 \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } d = \begin{bmatrix} 2 \\ 2 \\ 0.8 \\ 1.6 \end{bmatrix}$$

Constraints on input:

$$u_0 \le 2$$
$$-u_0 \le 1$$
$$u_1 \le 2$$
$$-u_1 \le 1$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \le \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad E = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

(Of course, any other C, d, E and f are allowed, provided the respective rows of (C and d) and (E and f) match.)

(c) $CX \le d \Leftrightarrow C(\Phi x(k) + \Gamma u) \le d$

$$\Leftrightarrow C\Phi x(k) + C\Gamma u \le d$$

$$\Leftrightarrow C\Gamma u \le d - C\Phi x(k)$$

$$\Rightarrow \quad \begin{aligned} C\Gamma u &\le d - C\Phi x(k) \\ Eu &\le f \end{aligned}$$

4c cont~)

$$\iff \begin{bmatrix} C\Gamma \\ E \end{bmatrix} u \leq \begin{bmatrix} d \\ f \end{bmatrix} + \begin{bmatrix} -C\Phi \\ 0 \end{bmatrix} x(k)$$

$$\Longrightarrow \quad G = \begin{bmatrix} C\Gamma \\ E \end{bmatrix}$$

$$h = \begin{bmatrix} d \\ f \end{bmatrix}$$

$$L = \begin{bmatrix} -C\Phi \\ 0 \end{bmatrix}$$

$$\Longrightarrow \quad h = \begin{bmatrix} 2 \\ 2 \\ 0.8 \\ 1.6 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Now, $C\Gamma = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 2 & 1 \\ -2 & -1 \end{bmatrix}$

and $C\Phi = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ -4 \end{bmatrix}$

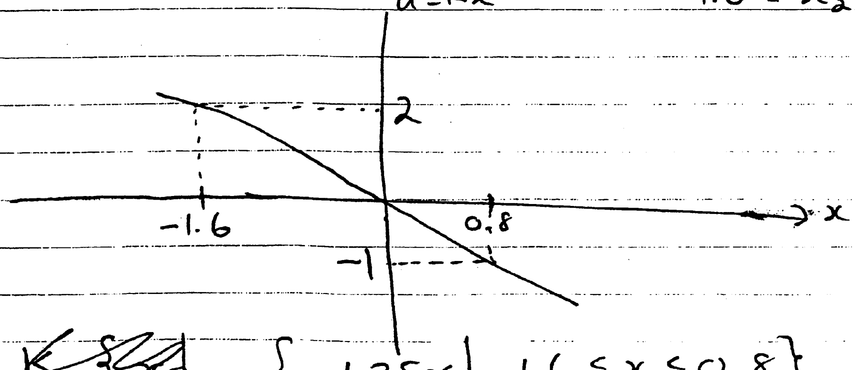
4c cont...)

$$\Rightarrow \quad G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 2 & 1 \\ -2 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

and

$$L = \begin{bmatrix} -2 \\ +2 \\ -4 \\ +4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4d) Single state, single input: $u = Kx$ Terminal constraint is $-1.6 \le x_2 \le 0.8$



K ~~$\{x|$~~ $\{-1.25x \mid -1.6 \le x \le 0.8\}$

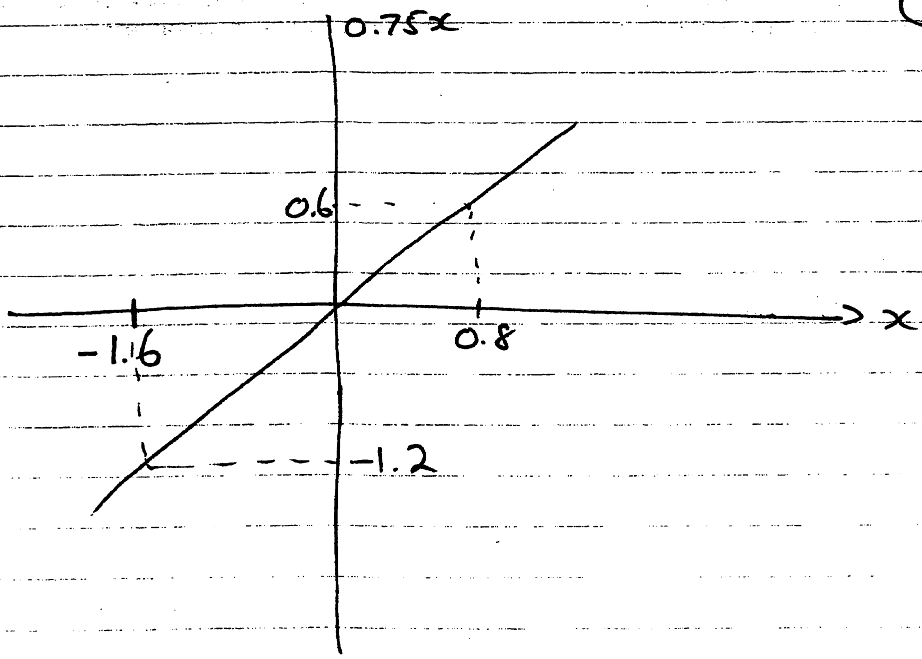
$= \{u \mid -1 \le u \le 2\}$ ~~Terminal~~

4d cont.) $\Rightarrow$ terminal constraint maps to inputs in the range $[-1, 2]$ under the control $u = -1.25x$.

$\Rightarrow$ terminal constraint input admissible because $u \in [-1, 2]$ is the constraint on the input.

4e) A set S is invariant if $(A+BK)x \in S \;\forall x \in S$.
(equivalently, $(A+BK)S \subseteq S$)

$A + BK = 2 - 1.25 = 0.75$ $\quad$ ($A = 2, B = 1$, $K = -1.25$)



The set $S = [-1.6, 0.8]$ maps to $[-1.2, 0.6]$ under the mapping $A + BK$, hence $(A+BK)S \subseteq S$

Since S is the terminal constraint, it follows that it is invariant for the closed-loop system $x(k+1) = (2+K)x(k)$

4f) Yes, the RHC law is feasible for all time.

A sufficient condition for a RHC law to be invariant for all time, is for the terminal constraint to be controlled invariant

In part (d), we showed that the ~~term~~ control law $u = Kx$ satisfies the input constraints for all x in the terminal constraint

In part (e) we showed that the terminal constraint is invariant under the ~~closed~~ control law $u = Kx$

⇒ The ~~observations~~ conclusions from parts (d) and (e) are sufficient to allow us to conclude that the terminal constraint is controlled invariant.

Module 4F3, Nonlinear and Predictive Control
Answers to 2004 exam

1. (a) — (b)(i) $x_1 = 0$, $x_2 = 0$. (b)(ii) — (b)(iii) Yes. (c) —

2. (a) — (b) 0 (c) 1 (d) $k > 2$, No.
   (e) Circle criterion: $k < \frac{1}{2}$, Popov criterion: $k < 2$.

3. —

4. (a) —
   (b) For example:

$$C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad d = \begin{bmatrix} 2 \\ 2 \\ 0.8 \\ 1.6 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

   (c)

$$G = \begin{bmatrix} C\Gamma \\ E \end{bmatrix}, \quad h = \begin{bmatrix} d \\ f \end{bmatrix}, \quad L = \begin{bmatrix} -C\Phi \\ 0 \end{bmatrix}$$

   (d) —
   (e) Yes.
   (f) Yes.