ENGINEERING TRIPOS PART IIB

Worked solutions - May 2004

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

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1 Consider the following signal $\{u(n)\}$ given by

$$u(n) = \alpha u(n-1) + v(n)$$

where $|\alpha| < 1$ and $\{v(n)\}$ is zero-mean white noise of variance $E\left(v^2(n)\right) = \sigma_v^2$.

This signal is filtered through a linear filter having impulse response $\{\beta(i)\}_{i=0}^{L-1}$ and the observations are

$$y(n) = \sum_{i=0}^{L-1} \beta(i)\, u(n-i) + w(n)$$

where $\{w(n)\}$ is zero-mean white noise of variance $E\left(w^2(n)\right) = \sigma_w^2$. The noise $\{v(n)\}$ is uncorrelated with $\{w(n)\}$.

(a) Consider the vector $\mathbf{u}(n) = (u(n)\ u(n-1)\ \ldots\ u(n-M+1))^{\mathrm{T}}$. Derive expressions for $E\left(\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right)$ and $E(y(n)\,\mathbf{u}(n))$, where $M > L$.

[50%]

Answer. One has

$$\begin{aligned} E\left\{u^2(n)\right\} &= E\left\{(\alpha u(n-1) + v(n))^2\right\} \\ &= \alpha^2 E\left\{u^2(n-1)\right\} + \sigma_v^2 \end{aligned}$$

thus

$$E\left\{u^2(n)\right\} = \frac{\sigma_v^2}{1-\alpha^2}$$

and

$$\begin{aligned} E\{u(n)\,u(n-1)\} &= E\{u(n-1)(\alpha u(n-1) + v(n))\}. \\ &= \alpha E\left\{u^2(n-1)\right\} = \frac{\alpha\sigma_v^2}{1-\alpha^2}. \end{aligned}$$

Similarly, one can easily show that

$$E\{u(k)\,u(l)\} = \frac{\alpha^{|k-l|}\sigma_v^2}{1-\alpha^2}.$$

One has

$$E\{y(n)\,\mathbf{u}(n)\} = (E(y(n)\,u(n))\ \ E(y(n)\,u(n-1))\ \ \ldots\ \ E(y(n)\,u(n-M+1)))^{\mathrm{T}}$$

(cont.

where

$$E\left(y\left(n\right)u\left(n-j\right)\right)=E\left(\left(\sum_{i=0}^{L-1}\beta\left(i\right)u\left(n-i\right)+v\left(n\right)\right)u\left(n-j\right)\right)$$

$$=\sum_{i=0}^{L-1}\beta\left(i\right)E\left(u\left(n-i\right)u\left(n-j\right)\right)+E\left(v\left(n\right)u\left(n-j\right)\right).$$

$$=\sum_{i=0}^{L-1}\beta\left(i\right)E\left(u\left(n-i\right)u\left(n-j\right)\right).$$

(b) Give the explicit expression for the Wiener filter $\mathbf{h}_{\text{opt}}$ that minimises

$$J\left(\mathbf{h}\right)=E\left(\left(y\left(n\right)-\mathbf{h}^{\mathrm{T}}\mathbf{u}\left(n\right)\right)^{2}\right)$$

and compute $J\left(\mathbf{h}_{\text{opt}}\right)$.

Hint. No additional calculation is required. [25%]

Answer. We know that the filter minimizing the error is the Wiener filter

$$\mathbf{h}_{\text{opt}}=\left(E\left\{\mathbf{u}\left(n\right)\mathbf{u}^{\mathrm{T}}\left(n\right)\right\}\right)^{-1}E\left\{y\left(n\right)\mathbf{u}\left(n\right)\right\}.$$

However, clearly $\mathbf{h}_{\text{opt}}=\left(\beta\left(0\right)\ldots\beta\left(L-1\right)\ 0\ldots0\right)$ minimizes the square error and in this case

$$J\left(\mathbf{h}_{\text{opt}}\right)=E\left(\left(y\left(n\right)-\mathbf{h}_{\text{opt}}\mathbf{u}\left(n\right)\right)^{2}\right)=E\left(w^{2}\left(n\right)\right)=\sigma_{w}^{2}.$$

(c) In a real-world environment, the filter $\left\{\beta\left(i\right)\right\}_{i=0}^{L-1}$ is unknown so the Wiener filter cannot be implemented. Describe a LMS algorithm to approximate the Wiener filter. Would you recommend the use of the LMS algorithm if $|\alpha|\simeq1$? Explain your answer, giving potential alternatives if you would not recommend LMS. [25%]

Answer. One has

$$\mathbf{h}\left(n\right)=\mathbf{h}\left(n-1\right)+\mu\left(y\left(n\right)-\mathbf{h}^{\mathrm{T}}\left(n-1\right)\mathbf{u}\left(n\right)\right)\mathbf{u}\left(n\right).$$

To ensure stability of the LMS, we require

$$\mu<\frac{1}{ME\left\{u^{2}\left(n\right)\right\}}$$

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where

$$E\left\{u^2(n)\right\} = \frac{\sigma_v^2}{1-\alpha^2}.$$

As $|\alpha| \to 1$, the input signal is more and more colored and to ensure stability μ has to be very small. Consequently, one should not use the LMS algorithm in this case. As an alternative, one can use the Normalized LMS or RLS.

2 (a) Consider the following recursive algorithm (whitened gradient search method)

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu\mathbf{R}^{-1}(\mathbf{p} - \mathbf{R}\mathbf{h}(n-1)) \qquad (1)$$

where $\mathbf{R}$ and $\mathbf{p}$ are respectively a definite positive matrix and a vector of appropriate dimensions. Assuming $\mathbf{h}(n)$ converges towards a limit $\mathbf{h}_{\text{opt}}$, find an expression for $\mathbf{h}_{\text{opt}}$. [25%]

Answer. If $\mathbf{h}(n)$ converges towards $\mathbf{h}_{\text{opt}}$, then

$$\mathbf{h}_{\text{opt}} = \mathbf{h}_{\text{opt}} + \mu\mathbf{R}^{-1}\left(\mathbf{p} - \mathbf{R}\mathbf{h}_{\text{opt}}\right).$$

It follows that

$$\mathbf{p} - \mathbf{R}\mathbf{h}_{\text{opt}} = 0 \Leftrightarrow \mathbf{h}_{\text{opt}} = \mathbf{R}^{-1}\mathbf{p}.$$

(b) Using (1) and the expression for $\mathbf{h}_{\text{opt}}$, obtain a recursion for

$$\mathbf{v}(n) = \mathbf{h}(n) - \mathbf{h}_{\text{opt}}.$$

Deduce a condition on μ ensuring convergence of the recursive algorithm whatever initial vector $\mathbf{v}(0)$ is chosen. [30%]

Answer. One has

$$\begin{aligned}\mathbf{h}(n) &= \mathbf{h}(n-1) + \mu\mathbf{R}^{-1}\left(\mathbf{R}\mathbf{h}_{\text{opt}} - \mathbf{R}\mathbf{h}(n-1)\right) \\ &= \mathbf{h}(n-1) + \mu\left(\mathbf{h}_{\text{opt}} - \mathbf{h}(n-1)\right) \\ &\Rightarrow \mathbf{v}(n) = (1-\mu)\,\mathbf{v}(n-1).\end{aligned}$$

It follows straightforwardly that the algorithm is stable if and only if $\mu < 1$.

(c)
Let $\mathbf{R}$ and $\mathbf{p}$ be given by $\mathbf{R} = E\left\{\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right\}$ and $\mathbf{p} = E\{\mathbf{u}(n)\,d(n)\}$ where $\mathbf{u}(n) = (u(n)\;\; u(n-1)\;\; \ldots\;\; u(n-M+1))^{\mathrm{T}}$. In practice, $\mathbf{R}$ and $\mathbf{p}$ are typically unknown and one only has access to a realisation of the input signal $\{u(n)\}$ and of the reference signal $\{d(n)\}$. The signals $\{u(n)\}$ and $\{d(n)\}$ are assumed stationary and ergodic. We consider the following recursive algorithm

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu\left[\mathbf{R}(n)\right]^{-1}\left(d(n) - \mathbf{u}^{\mathrm{T}}(n)\,\mathbf{h}(n-1)\right)\mathbf{u}(n) \qquad (2)$$

(CONTINUED OVER.

where

$$\mathbf{R}(n) = \left(1 - \frac{1}{n}\right)\mathbf{R}(n-1) + \frac{1}{n}\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n). \tag{3}$$

Explain why (2)-(3) can be interpreted as a stochastic gradient approximation of (1). [20%]

Answer. As the signal $\{u(n)\}$ is stationary and ergodic then

$$\mathbf{R}(n) = \frac{1}{n}\sum_{k=1}^{n}\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n) \rightarrow \mathbf{R}.$$

Moreover, one has

$$E\left[\left(d(n) - \mathbf{u}^{\mathrm{T}}(n)\,\mathbf{h}(n-1)\right)\mathbf{u}(n)\right] = -\nabla_{\mathbf{h}(n-1)}J(\mathbf{h})$$

where

$$J(\mathbf{h}) = E\left[\left(d(n) - \mathbf{u}^{\mathrm{T}}(n)\,\mathbf{h}\right)^{2}\right].$$

Thus the algorithm defined by (2)-(3) corresponds to a stochastic gradient algorithm minimizing $\mathbf{R}^{-1}J(\mathbf{h})$.

(d) Assume now that the signal $\{u(n)\}$ is not stationary. Explain why the algorithm defined by (2)-(3) would not be useful in this context. Suggest a modification that allows for a non-stationary environment. [25%]

Answer. In the non-stationary context, this algorithm would be inefficient as

$$\mathbf{R}(n) = \frac{1}{n}\sum_{k=1}^{n}\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)$$

does not converge towards any meaningful quantity. In this case, it would be more suitable to define

$$\mathbf{R}(n) = \lambda\mathbf{R}(n-1) + (1-\lambda)\,\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)$$

where λ is a suitable forgetting factor; $0 < \lambda < 1$.

3 (a) Describe the parametric approach to power spectrum estimation. Your discussion should include the ARMA, AR and MA models and a comparison of parametric methods with non-parametric methods such as the periodogram. [30%]

Answer:

Bookwork as follows from lecture notes (more detailed than required):

- Periodogram-based methods can lead to biased estimators with large variance

- If the physical process which generated the data is known or can be well approximated, then a parametric model can be constructed

- Careful estimation of the parameters in the model can lead to power spectrum estimates with improved bias/variance.

- We will consider spectrum estimation for LTI systems driven by a white noise input sequence.

- If a random process $\{X_n\}$ can be modelled as white noise exciting a filter with frequency response $H(e^{j\omega T})$ then the spectral density of the data can be expressed as:

$$\boxed{S_X(e^{j\omega T}) = \sigma_w^2 \, |H(e^{j\omega T})|^2}$$

where σ_w^2 is the variance of the white noise process. [It is usually assumed that $\sigma_w^2 = 1$ and the scaling is incorporated as gain in the frequency response]

- We will study models in which the frequency response $H(e^{j\omega T})$ can be represented by a finite number of parameters which are estimated from the data.

- Parametric models need to be chosen carefully - an inappropriate model for the data can give misleading results

ARMA Models A quite general representation is the autoregressive moving-average (ARMA) model:

•The ARMA(P,Q) model difference equation representation is:

$$\boxed{x_n = -\sum_{p=1}^{P} a_p\, x_{n-p} + \sum_{q=0}^{Q} b_q\, w_{n-q}} \tag{4}$$

where:

a_p are the AR parameters,

b_q are the MA parameters

and $\{W_n\}$ is a zero-mean stationary white noise process with unit variance, $\sigma_w^2 = 1$.

•Clearly the ARMA model is a pole-zero IIR filter-based model with transfer function

$$H(z) = \frac{B(z)}{A(z)}$$

where:

$$A(z) = 1 + \sum_{p=1}^{P} a_p z^{-p}, \quad B(z) = \sum_{q=0}^{Q} b_q z^{-q}$$

With $Q = 0$ we have the AR model and with $P = 0$ the MA model.

•Unless otherwise stated we will always assume that the filter is stable, i.e. the poles (solutions of $A(z) = 0$) all lie *within* the unit circle (we say in this case that $A(z)$ is *minimum phase*). Otherwise the autocorrelation function is undefined and the process is technically *non-stationary.*

•Hence the power spectrum of the ARMA process is:

$$S_X(e^{j\omega T}) = \frac{|B(e^{j\omega T})|^2}{|A(e^{j\omega T})|^2}$$

The ARMA model is quite a flexible and general way to model a stationary random process:

•The poles model well the *peaks* in the spectrum (sharper peaks implies poles closer to the unit circle)

•The zeros model troughs in the spectrum

(cont.

•Complex spectra can be approximated well by large model orders P and Q

Note however, that model order determination is critical for ARMA modelling and an ARMA model may not be appropriate for certain datasets.

Compared with non-parametric methods, the variance in estimation could be lower, particularly if the model is appropriate for the data, but if not then an (asymptotically) unbiased non-parametric method such as the periodogram may be a better choice.

(b) Show that the autocorrelation function $R_{XX}[k]$ for a Q th order moving average (MA) model can be expressed as

$$R_{XX}[k] = c_k, \quad k = 0, 1, ..., Q$$

where the terms c_k should be carefully expressed in terms of the moving average parameters $\{b_q\}$. [30%]

Answer:

This may be adapted from the full ARMA proof in the lecture notes, as follows.

The autocorrelation function $R_{XX}[r]$ for the output x_n of the MA model is:

$$R_{XX}[r] = E[x_n x_{n+r}]$$

Substituting for x_{n+r} from the model equation gives:

$$R_{XX}[r] = E\left[x_n \left\{\sum_{q=0}^{Q} b_q\, w_{n+r-q}\right\}\right]$$

$$= \sum_{q=0}^{Q} b_q\, E[x_n w_{n+r-q}]$$

The white noise process $\{W_n\}$ is wide-sense stationary so that $\{X_n\}$ is also wide-sense stationary. Therefore:

$$\sum_{q=0}^{Q} b_q\, R_{XW}[r-q] \qquad (5)$$

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The cross-correlation term $R_{XW}[.]$ can be obtained as follows. Let the system impulse response be $h_n = b_n$, since the MA model is an FIR filter, then:

$$x_n = \sum_{m=0}^{Q} b_m \, w_{n-m}$$

Therefore,

$$E[x_n \, w_{n+k}] = E[w_{n+k} \sum_{m=0}^{Q} b_m \, w_{n-m}]$$

$$R_{XW}[k] = \sum_{m=0}^{Q} b_m \, E[w_{n+k} \, w_{n-m}]$$

Now the noise is a zero-mean stationary white process so that:

$$E[w_{n+k} \, w_{n-m}] = \begin{cases} \sigma_W^2 & \text{if } m = -k \\ 0 & \text{otherwise} \end{cases}$$

and $\sigma_W^2 = 1$ without loss of generality. Hence,

$$R_{XW}[k] = h_{-k}$$

Substituting this expression for $R_{XW}[k]$ into equation 5 gives the *Yule-Walker Equation* for an MA process,

$$\boxed{R_{XX}[r] = \sum_{q=0}^{Q} b_q \, b_{q-r}} = c_r \tag{6}$$

and c_r may be simplified as

$$c_r = \begin{cases} \sum_{q=r}^{Q} b_q \, h_{q-r} & \text{if } r \leq Q \\ 0 & \text{if } r > Q \end{cases} \tag{7}$$

(c) Three values for the autocorrelation function of a random process are determined as:

$$R_{XX}[0] = 1.00, \;\; R_{XX}[1] = -0.48, \;\; R_{XX}[2] = 0.19$$

(cont.

(up to two decimal places).

Fit a minimum phase MA model with $Q = 2$ to this data using the spectral factorisation method, carefully explaining the steps in your working. [40%]

You may use the following factorisation to assist in answering part (c):

$$z^{-2} - 2.5z^{-1} + 5.25 - 2.5z + z^2 =$$
$$z^{-2}(z - 0.5\exp(i\pi/3))(z - 0.5\exp(-i\pi/3))(z - 2\exp(i\pi/3))(z - 2\exp(-i\pi/3))$$

Answer:

From lectures we have that:

$$\boxed{B(z)\,B(z^{-1}) = \sum_{r=-Q}^{Q} R_{XX}[r]\,z^{-r}} \qquad (8)$$

Hence we can identify (from the given factorisation) the zeros of $B(z)$ as $0.5\exp(i\pi/3)$ and $0.5\exp(-i\pi/3)$. Multiplying this out gives:

$$(1 - z^{-1}0.5\exp(i\pi/3))(1 - z^{-1}0.5\exp(-i\pi/3)) = 1 - 0.5z^{-1} + 0.25z^{-2}$$

Comparing with the measured autocorrelation parameter $R_{XX}[0] = 1$ we have:

$$1 = \sum_{i=0}^{2} b_i^2 = g^2(1 + 0.25 + 0.0625) = 1.3125\,g^2$$

Hence

$$B(z) = \frac{1}{\sqrt{1.3125}}(1 - 0.5z^{-1} + 0.25z^{-2})$$

and the MA model is:

$$x_n = 0.873(u_n - 0.5u_{n-1} + 0.25u_{n-2})$$

where u_n is white noise with variance 1.

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4 (a) Discuss the effects of time-domain windowing in the spectrum analysis of discrete-time signals. You should include a discussion of spectral leakage, spectral smearing and contrast the properties of Hamming, Hanning and rectangular windows. [40%]

Answer:

Bookwork from lecture notes, as follows (more detailed than required):

Consider the discrete case shown in figure 1.

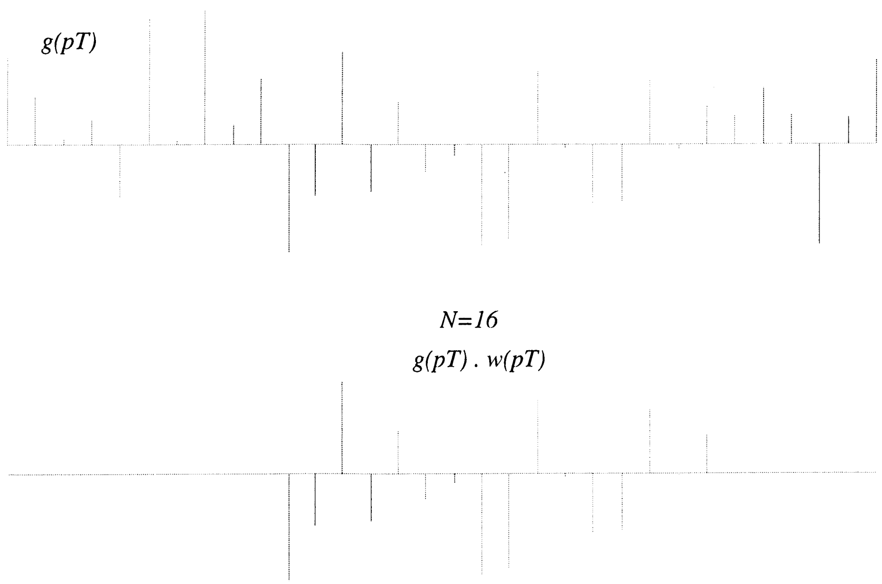

Fig. 1

The sampled values of the window signal are $w_p = w(pT)$ and $g_p = g(pT)$, respectively.

(cont.

Now, Take the DTFT of the windowed signal $w_p g_p$:

$$
\begin{aligned}
G(e^{j\omega T}) &= \sum_{p=-\infty}^{\infty} g_p \, e^{-jp\omega T} \\
G_w(e^{j\omega T}) &= \sum_{p=-\infty}^{\infty} \{g_p \, w_p\} \, e^{-jp\omega T} \\
&= \sum_{p=-\infty}^{\infty} g_p \left\{ \frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta}) e^{jp\theta} \, d\theta \right\} e^{-jp\omega T} \\
&= \frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta}) \sum_{p=-\infty}^{\infty} g_p \, e^{-jp(\omega T-\theta)} \, d\theta \\
G_w(e^{j\omega T}) &= \frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta}) \, G(e^{j(\omega T-\theta)}) \, d\theta
\end{aligned}
$$

Exactly as before, the spectrum of the windowed signal is the convolution of the infinite duration signal spectrum and the window spectrum.

Note that all discrete time spectra are periodic functions of frequency.

As for the continuous case we can consider the use of tapered windows and one class of window functions is the *generalised Hamming window* given by

$$w_n = \alpha - (1-\alpha)\cos\left(\frac{2\pi}{N}n\right) \quad \text{for } n = 0 \text{ to } N-1$$

$\alpha = 1$ Rectangular window

$\alpha = 0.5$ Hanning window (Raised Cosine or Cosine Arch)

$\alpha = 0.54$ Hamming window

We can evaluate the spectrum (DTFT) of the generalised window as follows

$$W(e^{j\omega T}) = \sum_{p=0}^{N-1} \left\{\alpha - \frac{(1-\alpha)}{2}\left[e^{j\frac{2\pi}{N}p} + e^{-j\frac{2\pi}{N}p}\right]\right\} e^{-jp\omega T}$$

which gives

$$W(e^{j\omega T}) = e^{-j(N-1)\frac{\omega T}{2}} \left\{\alpha \frac{\sin(N\frac{\omega T}{2})}{\sin(\frac{\omega T}{2})} \right.$$
$$\left. + \frac{1-\alpha}{2}\left[e^{-j\frac{\pi}{N}} \frac{\sin\left[\frac{N}{2}(\omega T - \frac{2\pi}{N})\right]}{\sin\left[\frac{1}{2}(\omega T - \frac{2\pi}{N})\right]} + e^{j\frac{\pi}{N}} \frac{\sin\left[\frac{N}{2}(\omega T + \frac{2\pi}{N})\right]}{\sin\left[\frac{1}{2}(\omega T + \frac{2\pi}{N})\right]}\right]\right\}$$

This is shown in figure 2 for the Hanning window $(\alpha = 0.5)$ and in figure 3 for other values of α .

Note that many other classes with different side-lobe and central lobe properties are available, e.g. Blackman, Bartlett, Chebyshev, Kaiser, ... These are all available as Matlab functions, so you can easily display them and their DFT within Matlab.

Matlab demo:

`disc_wind.m`

Type `window` at the Matlab command line for an interactive window display program.

(b) The periodogram estimate for the power spectrum estimate of a random process can be expressed as:

$$\hat{S}_X(e^{j\omega T}) = \sum_{k=-(N-1)}^{N-1} \hat{R}_{XX}[k]\, e^{-jk\omega T}$$

where $\hat{R}_{XX}[k]$ is the estimated autocorrelation sequence, given by:

$$\hat{R}_{XX}[k] = \frac{1}{N}\sum_{n=0}^{N-1-k} x_n\, x_{n+k} \qquad 0 \le k < N$$

(cont.

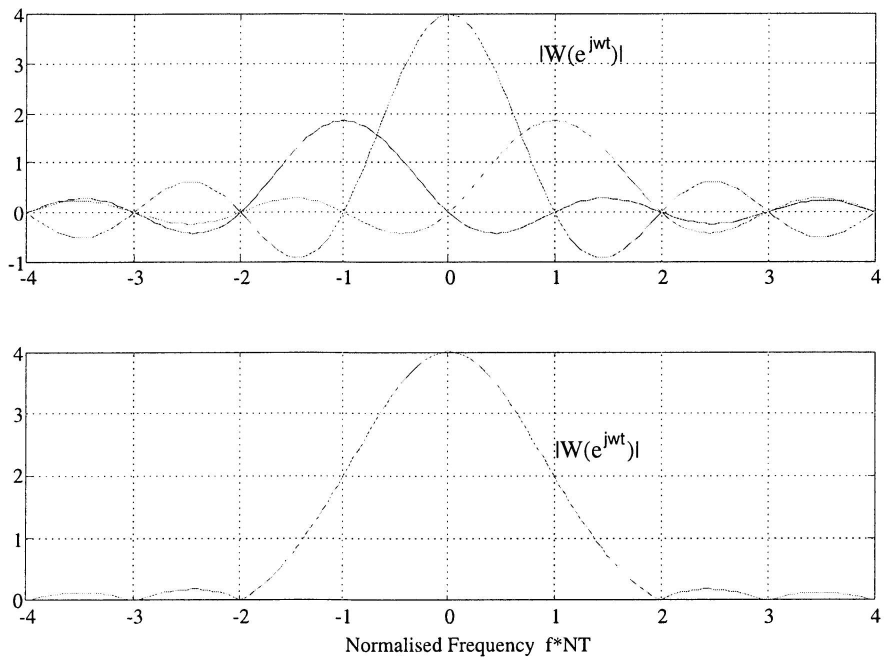

Fig. 2

and

$$\hat{R}_{XX}[k] = \hat{R}_{XX}[-k], \qquad -N < k < 0$$

Show that $\hat{R}_{XX}[k]$ is a biased estimate of the autocorrelation function, and hence show that the expected value of the periodogram is given by:

$$E[\hat{S}_X(e^{j\omega T})] = \sum_{k=-\infty}^{\infty} w_k R_{XX}[k]\, e^{-jk\omega T}$$

where w_k is the Bartlett window of length (2N), defined as:

$$w_k = \begin{cases} \frac{N-|k|}{N}, & |k| < N \\ 0, & \text{otherwise} \end{cases}$$

[30%]

Answer: Again, this can be done with modifications of the lecture notes, as follows (we require the biased version of R_{XX} in this case):

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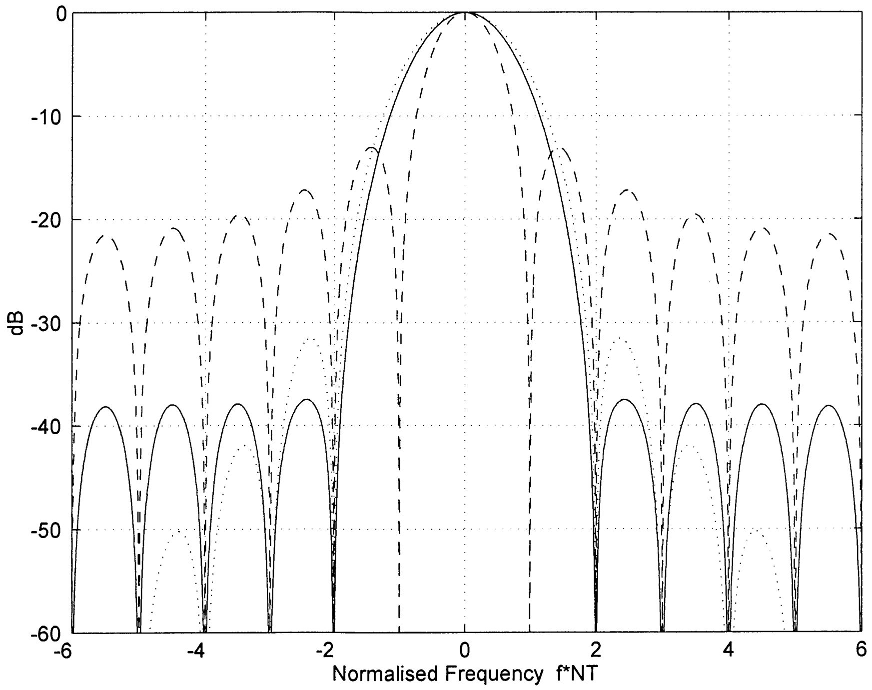

Fig. 3

(cont.

•The expected value of the periodogram may be written as:

$$E[\hat{S}_X(e^{j\omega T})] = E\left[\sum_{k=-(N-1)}^{N-1} \hat{R}_{XX}[k]\, e^{-jk\omega T}\right]$$
$$= \sum_{k=-(N-1)}^{N-1} E[\hat{R}_{XX}[k]]\, e^{-jk\omega T} \qquad (9)$$

i.e. the DTFT of the expected autocorrelation function estimate

•What is $E[\hat{R}_{XX}[k]]$?

$E[\hat{R}_{XX}[k]]$:

(i) Using the given form of autocorrelation function estimate:

$$E[\hat{R}_{XX}[k]] = E\left[\frac{1}{N}\sum_{n=0}^{N-1-k} x_n\, x_{n+k}\right]$$
$$= \frac{1}{N}\sum_{n=0}^{N-1-k} E[x_n x_{n+k}]$$
$$= \frac{1}{N}\sum_{n=0}^{N-1-k} R_{XX}[k]$$
$$= \frac{N-k}{N} R_{XX}[k]$$

hence biased as required.

We can summarize this results more conveniently (noting that $\hat{R}_{XX}[-k] = \hat{R}_{XX}[k]$) as:

$$E[\hat{R}_{XX}[k]] = w_k R_{XX}[k], \qquad k = -N+1, ..., N-1$$

where,

$$w_k = \begin{cases} \frac{N-|k|}{N}, & |k| < N \\ 0, & \text{otherwise} \end{cases} \quad \text{(Bartlett or triangular window)}$$

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•Substituting into the expression for $E[\hat{S}_X(e^{j\omega T})]$ we obtain:

$$\begin{aligned} E[\hat{S}_X(e^{j\omega T})] &= \sum_{k=-(N-1)}^{N-1} E[\hat{R}_{XX}[k]]\, e^{-jk\omega T} \\ &= \sum_{k=-(N-1)}^{N-1} w_k R_{XX}[k]\, e^{-jk\omega T} \\ &= \sum_{k=-(\infty)}^{\infty} w_k R_{XX}[k]\, e^{-jk\omega T} \end{aligned}$$

(ii) The two dominant frequency components in a random process are at frequencies ω_1 rad/s and ω_2 rad/s. The sampling period is $T = 0.1s$. Determine approximately the minimum window length T required to resolve the two frequency components in the power spectrum estimate, if they are likely to be spaced as little as 0.02π rad/s apart. State clearly any assumptions you make. You may assume that the normalised 3dB bandwidth of a Bartlett window having length M samples is $1.28(2\pi/M)$ rad and its 6dB bandwidth is $1.78(2\pi/M)$ rad. [30%]

Answer: The effect of the periodogram method in the frequency domain is to convolve the true power spectrum's frequency components (delta functions) with the Bartlett window. Hence two components are approximately resolved if their half-power (3dB) bands do not overlap, i.e. they are spaced $2 * 1.28(2\pi/(2N))$ apart (note: the Bartlett window in the periodogram estimate is of length $M = 2N$). Thus we have:

$$\Delta(\omega T) \geq 0.2\pi * 0.1$$

which must equal twice the 3dB bandwidth of the shortest appropriate Bartlett window, i.e. :

$$2 * 1.28 * 2\pi/(2N) \geq 0.2\pi * 0.1$$

Hence

$$N \geq 2 * 1.28/0.1 * 0.2 = 128$$

END OF PAPER