Module 4F8 – IMAGE PROCESSING AND IMAGE CODING – Solutions

1 (a) **Windowing Method**

Taking the inverse FT of some desired zero-phase 2d frequency response will, in general, produce a filter with infinite support. One method of producing a filter with *finite* support is via the *windowing* method. In this, we simply multiply the infinite support filter by a window function $w(n_1, n_2)$ which forces the impulse response coefficients, $h(n_1, n_2)$, to zero for $n_1, n_2 \notin R_h$ where R_h is the desired support region.

If $H_d(\omega_1, \omega_2)$ is the desired frequency response and $h_d(n_1, n_2)$ the corresponding impulse response, then the windowed filter is given by

$$h(n_1, n_2) = h_d(n_1, n_2)\; w(n_1, n_2)$$

The frequency response, $H(\omega_1, \omega_2)$, of this filter is then clearly the convolution of the desired frequency response and the FT of the window function, $W(\omega_1, \omega_2)$.

ie:

$$H(\omega_1, \omega_2) = H_d(\omega_1, \omega_2) \otimes W(\omega_1, \omega_2)$$

where $\otimes$ represents convolution.

The effect of the window is therefore to smooth H_d.

Two methods of forming 2d window functions from 1d window functions are:

i) **Product of 1d windows**

$$w(n_1, n_2) = w_1(n_1)\; w_2(n_2) \qquad \text{or} \qquad w(u_1, u_2) = w_1(u_1)\; w_2(u_2)$$

eg. if $w_i(u_i) = \begin{cases} 1 & \text{if } |u_i| < U_i \\ 0 & \text{otherwise} \end{cases}$

then $w(u_1, u_2) = w_1(u_1)\; w_2(u_2) = \begin{cases} 1 & \text{if } |u_1| < U_1 \;\&\; |u_2| < U_2 \\ 0 & \text{otherwise} \end{cases}$

ii) **Rotation of 1d windows**

$$w(u_1, u_2) = w_1(u)\big|_{u = \sqrt{u_1^2 + u_2^2}}$$

ie. we rotate a 1d continuous window to form a 2d continuous window (which is then sampled to produce a discrete window). [25%]

(b) **Hamming-type Windows**

$$w_1(u_1) = \begin{cases} \alpha + \beta \cos \dfrac{\pi u_1}{U_1} & \text{if } |u_1| < U_1 \\ 0 & \text{otherwise.} \end{cases}$$

The 2d 'Hamming' window formed by the product of two such 1d windows is

$$w(u_1, u_2) = w_1(u_1) w_2(u_2) = \begin{cases} \left\{\alpha + \beta \cos \dfrac{\pi u_1}{U_1}\right\} \left\{\alpha + \beta \cos \dfrac{\pi u_2}{U_2}\right\} & \text{if } \begin{array}{l} |u_1| < U_1 \\ |u_2| < U_2 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

Now $W(\omega_1, \omega_2) = W_1(\omega_1) W_2(\omega_2)$ as the FT is also separable.

$$\begin{aligned} \therefore W_1(\omega_1) &= \int_{-U_1}^{U_1} \left(\alpha + \beta \cos \frac{\pi u_1}{U_1}\right) e^{-j\omega_1 u_1} \, du_1 \\ &= \int_{-U_1}^{U_1} \alpha \, e^{-j\omega_1 u_1} \, du_1 + \int_{-U_1}^{U_1} \beta \cos \frac{\pi u_1}{U_1} \, e^{-j\omega_1 u_1} \, du_1 \end{aligned}$$

Using the data book FT table for the first integral and the formula given in the question paper for the second one gives:

$$\begin{aligned} W_1(\omega_1) &= 2\alpha U_1 \, \text{sinc}(\omega_1 U_1) + \beta \, \frac{2\, \omega_1 \, U_1^2}{\pi^2 - \omega_1^2 U_1^2} \, \sin(\omega_1 U_1) \\ &= 2U_1 \left\{\alpha + \frac{\beta \, \omega_1^2 \, U_1^2}{\pi^2 - \omega_1^2 U_1^2}\right\} \text{sinc}(\omega_1 \, U_1) \end{aligned}$$

Similarly

$$W_2(\omega_2) = 2U_2 \left\{\alpha + \frac{\beta \, \omega_2^2 \, U_2^2}{\pi^2 - \omega_2^2 U_2^2}\right\} \text{sinc}(\omega_2 \, U_2)$$

Hence

$$W(\omega_1, \omega_2) = f(\omega_1, U_1) \, f(\omega_2, U_2) \;\; \text{sinc}(\omega_1 U_1) \;\; \text{sinc}(\omega_2 U_2)$$

where

$$\boxed{f(\omega_i, U_i) = 2U_i \left\{\alpha + \frac{\beta \, \omega_i^2 \, U_i^2}{\pi^2 - \omega_i^2 U_i^2}\right\}}$$

For a **Hamming window**:

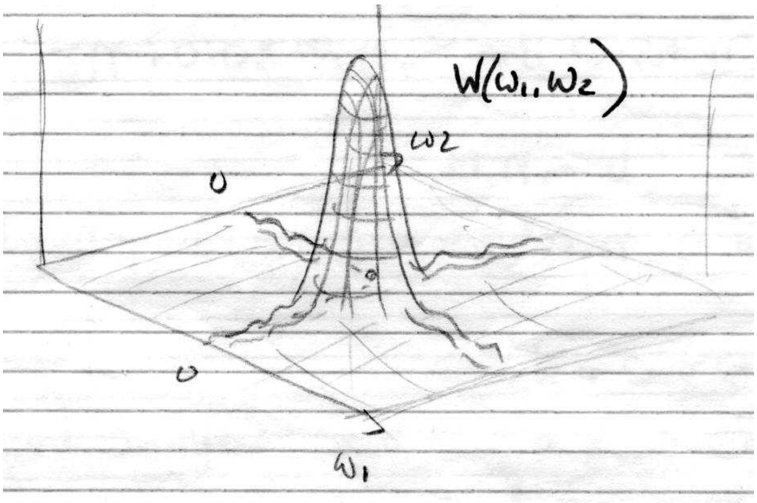
$$\alpha = 0.54 \qquad \text{and} \qquad \beta = 0.46$$

$$\therefore f(\omega, U) = 2U \left\{\frac{\alpha\pi^2 - (\alpha - \beta)\, \omega^2 \, U^2}{\pi^2 - \omega^2 U^2}\right\} = 2U \left\{\frac{0.54\pi^2 - 0.08 \, \omega^2 \, U^2}{\pi^2 - \omega^2 U^2}\right\}$$

For ω small: $f(\omega, U) \simeq 2\,U \times 0.54 = 1.08\,U$ (a constant)

So, for ω_1, ω_2 small, the spectrum follows a 2d sinc function – the sidelobes occur predominantly along the ω_1 and ω_2 axes and decay rapidly with increasing ω (due to the $\dfrac{1}{\pi^2 - \omega^2 U^2}$ factor).

Sketch of 2d Spectrum for a Hamming Window



[60%]

(c) **Properties of good window functions:**

Since the actual frequency response is the convolution of the desired frequency response with the spectrum of the window function, we would clearly like the spectrum of the window function to be as much like a delta-function as possible. We therefore want something with a small mainlobe width and sidelobes which are as small in amplitude as possible.

Forming 2d windows from 1d windows is a very simple way of producing 2d windows with predictable behaviour.

i) **Product of 1d windows** - spectrum of 2d window is simply the product of the spectra of the 1d windows - so easy to deal with. One problem is that sidelobes occur along the axes so that the resulting filter has inbuilt preferential directions (see sketch above).

ii) **Rotation method** - spectrum is not as easy to visualise and sampling is a little harder. One advantage of this method is that it produces windows with spectra which are rotationally symmetric. Hence sidelobes do not occur along particular directions. [15%]

2 (a) **Discrete convolution**

The continuous convolution, given, is:

$$y(u_1, u_2) = \int\int h(v_1, v_2)\, x(u_1 - v_1, u_2 - v_2)\, dv_1\, dv_2 + d(u_1, u_2)$$

Infinite limits for the integrations are assumed.

In discrete form we write the above image model equation as

$$y(n_1, n_2) = \sum_{m_2=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} h(m_1, m_2)\, x(n_1 - m_1, n_2 - m_2) + d(n_1, n_2)$$

where the m_i and n_i are integers.

In the absence of noise we assume there is no $d(n_1, n_2)$ term, and take the FT of the remaining convolution equation to give

$$Y(\omega_1, \omega_2) = H(\omega_1, \omega_2)\; X(\omega_1, \omega_2)$$

where $H(\omega_1, \omega_2) = \sum_{n_2} \sum_{n_1} h(n_1, n_2)\; e^{-j(\omega_1 n_1 + \omega_2 n_2)}$

$$\therefore X(\omega_1, \omega_2) = \frac{1}{H(\omega_1, \omega_2)}\; Y(\omega_1, \omega_2)$$

$\Rightarrow \dfrac{1}{H(\omega_1, \omega_2)}$ is the inverse filter that we can apply to the FT of our observations, Y, in order to recover X and hence the original image via

$$x(n_1, n_2) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2)\; e^{j(\omega_1 n_1 + \omega_2 n_2)}\, d\omega_1\, d\omega_2$$

If $H(\omega_1, \omega_2)$ has zeros or regions where it becomes very small, then we clearly have problems. In particular, small noise in regions of the frequency plane where $\frac{1}{H}$ is very large can be hugely amplified. In practice one can lessen the sensitivity to noise by thresholding the frequency response.

eg:

$$H_g(\omega_1, \omega_2) = \begin{cases} \dfrac{1}{H(\omega_1, \omega_2)} & \text{if } \dfrac{1}{H} < \gamma \\ 0 & \text{otherwise} \end{cases}$$

[35%]

b) **In vector notation**

$$y(\mathbf{n}) = \sum_{\mathbf{m} \in \mathcal{Z}^2} h(\mathbf{m})\, x(\mathbf{n}-\mathbf{m}) + d(\mathbf{n})$$

We want to find the $\hat{x}(\mathbf{n})$ that minimises the squared error

$$Q = E\left\{[x(\mathbf{n}) - \hat{x}(\mathbf{n})]^2\right\} = E\left\{\left[x(\mathbf{n}) - \sum_{\mathbf{q}} g(\mathbf{q})\, y(\mathbf{n}-\mathbf{q})\right]^2\right\}$$

Since we are told that, we look for an $\hat{x}(\mathbf{n})$ which can be written as

$$\hat{x}(\mathbf{n}) = \sum_{\mathbf{q}} g(\mathbf{q})\, \mathbf{y}(\mathbf{n}-\mathbf{q})$$

To find the minimum, we differentiate Q wrt $g(\mathbf{p})$:

$$\frac{\partial Q}{\partial g(\mathbf{p})} = E\left\{2\left[x(\mathbf{n}) - \sum_{\mathbf{q}} g(\mathbf{q})\, y(\mathbf{n}-\mathbf{q})\right][-\mathbf{y}(\mathbf{n}-\mathbf{p})]\right\} = 0 \qquad \forall\, \mathbf{p} \in \mathcal{Z}^2$$

Hence for the optimum filter $\hat{g}(\mathbf{q})$:

$$E\left\{x(\mathbf{n})\, y(\mathbf{n}-\mathbf{p})\right\} = \sum_{\mathbf{q}} \hat{g}(\mathbf{q})\, E\left\{y(\mathbf{n}-\mathbf{q})\, y(\mathbf{n}-\mathbf{p})\right\}$$

If the images are spatially stationary we have

$$E\left\{x(\mathbf{n})\, y(\mathbf{n}-\mathbf{p})\right\} = E\left\{x(\mathbf{n}-\mathbf{p}+\mathbf{p})\, y(\mathbf{n}-\mathbf{p})\right\} = E\left\{x(\mathbf{k}+\mathbf{p})\, y(\mathbf{k})\right\} = R_{yx}(\mathbf{p})$$

which is the cross-correlation between y and x.

Similarly

$$E\left\{y(\mathbf{n}-\mathbf{q})\, y(\mathbf{n}-\mathbf{p})\right\} = E\left\{y(\mathbf{k})\, \mathbf{y}(\mathbf{k}+\mathbf{q}-\mathbf{p})\right\} = R_{yy}(\mathbf{q}-\mathbf{p})$$

$$\therefore R_{yx}(\mathbf{p}) = \sum_{\mathbf{q}} \hat{g}(\mathbf{q})\, R_{yy}(\mathbf{q}-\mathbf{p}) \qquad \forall\, \mathbf{p} \in \mathcal{Z}^2$$

Taking the FT of this equation and letting $\mathbf{k} = \mathbf{q} - \mathbf{p}$:

$$\begin{aligned} P_{yx}(\boldsymbol{\omega}) &= \sum_{\mathbf{q}} \hat{g}(\mathbf{q}) \sum_{\mathbf{p}} R_{yy}(\mathbf{q}-\mathbf{p})\, e^{-j\boldsymbol{\omega}^T\mathbf{p}} \\ &= \sum_{\mathbf{q}} \hat{g}(\mathbf{q}) \sum_{\mathbf{k}} R_{yy}(\mathbf{k})\, e^{j\boldsymbol{\omega}^T\mathbf{k}}\, e^{-j\boldsymbol{\omega}^T\mathbf{q}} \\ &= \hat{G}(\boldsymbol{\omega})\, P_{yy}^*(\boldsymbol{\omega}) \end{aligned}$$

Since power spectra are purely real, $P^*_{yy}(\boldsymbol{\omega}) = P_{yy}(\boldsymbol{\omega})$.

$$\therefore \quad \hat{G}(\boldsymbol{\omega}) = \frac{P_{yx}(\boldsymbol{\omega})}{P_{yy}(\boldsymbol{\omega})} \qquad \text{as required.}$$

[50%]

c) **Wiener filter**

If we write the above filter, $\hat{G}(\boldsymbol{\omega})$, in terms of $H(\boldsymbol{\omega})$, $P_{xx}(\boldsymbol{\omega})$ and $P_{dd}(\boldsymbol{\omega})$, we obtain the standard form of the Wiener filter which is

$$\hat{G}(\boldsymbol{\omega}) = \frac{H^*(\boldsymbol{\omega})\ P_{xx}(\boldsymbol{\omega})}{|H(\boldsymbol{\omega})|^2\ P_{xx}(\boldsymbol{\omega}) + P_{dd}(\boldsymbol{\omega})}$$

We can see from this form that

$$\text{if} \qquad |H(\boldsymbol{\omega})|^2\ P_{xx}(\boldsymbol{\omega}) \gg P_{dd}(\boldsymbol{\omega}) \qquad \text{then} \qquad \hat{G}(\boldsymbol{\omega}) \rightarrow \frac{1}{H(\boldsymbol{\omega})}$$

which is simply the inverse filter we discussed in part (a).

Thus the Wiener filter tends to the inverse filter when the noise power spectrum P_{dd} is much smaller than the filtered signal power spectrum $|H|^2 P_{xx}$. [15%]

3 (a) **DCT matrix**

T is orthonormal if the dot product of any row with any other row of **T** is zero and if the norm of each row (square root of the sum of the squares of the elements) is unity.

Norm of 1st row and 3rd row $= \sqrt{4a^2} = 1$
Norm of 2nd row and 4th row $= \sqrt{2b^2 + 2c^2} = \sqrt{0.8536 + 0.1464} = 1$

1st row and 3rd row are even symmetric, while 2nd row and 4th row are odd symmetric, so dot products between even and odd rows are zero.
Dot product between 1st and 3rd rows $= 2a^2 - 2a^2 = 0$
Dot product between 2nd and 4th rows $= 2bc - 2cb = 0$

Hence **T** is orthonormal.

1-D transform on a column vector: $\mathbf{y} = \mathbf{T}\,\mathbf{x}$
1-D transform on columns of a matrix: $\mathbf{Y}_1 = \mathbf{T}\,\mathbf{X}$
1-D transform on rows of a matrix: $\mathbf{Y} = \mathbf{Y}_1\,\mathbf{T}^T$
So 2-D transform on rows and columns of a matrix: $\mathbf{Y} = \mathbf{T}\,\mathbf{X}\,\mathbf{T}^T$ [25%]

(b) **DCT of edge subimage**

Taking transforms of columns first (for simplicity):

$$\mathbf{Y}_1 = \mathbf{T}\,\mathbf{X} = \begin{bmatrix} 4ap & 4ap & 4ap & 4aq \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now transforming the rows of $\mathbf{Y}_1$

$$\begin{aligned} \mathbf{Y} \;=\; \mathbf{Y}_1\,\mathbf{T}^T \;&=\; \begin{bmatrix} 4ap & 4ap & 4ap & 4aq \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & a & c \\ a & c & -a & -b \\ a & -c & -a & b \\ a & -b & a & -c \end{bmatrix} \\ &=\; \begin{bmatrix} 4a^2(3p+q) & 4ab(p-q) & -4a^2(p-q) & 4ac(p-q) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

[25%]

(c) **Coefficient selection**

Putting in values for a, b, c, p, q:

$$\mathbf{Y} = \begin{bmatrix} 140 & -78.40 & 60 & -32.47 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the transform matrix is orthonormal, $\mathbf{T}^{-1} = \mathbf{T}^T$ and the energy of the signal is always the same as the energy of its transform coefficients (energy is preserved between signal and transform domains). This is because, if $\mathbf{y} = \mathbf{T}\mathbf{x}$, then for any $\mathbf{x}$:

$$\mathbf{y}^T \mathbf{y} = (\mathbf{T}\mathbf{x})^T \mathbf{T}\mathbf{x} = \mathbf{x}^T \mathbf{T}^T \mathbf{T}\mathbf{x} = \mathbf{x}^T \mathbf{I} \mathbf{x} = \mathbf{x}^T \mathbf{x}$$

Hence if any transform coefficients are set to zero, the energy of the reconstructed image is reduced by the energy of these coefficients. The resulting squared error (error energy) is equal to that of the coefficients that were set to zero because the transform basis functions are orthogonal to each other, so the squared error is just the sum of the squares of the amplitude changes in all the basis functions.

To minimize the squared error, we must set to zero those coefficents with minimum energy, which are 60 and -32.47 in the above matrix. Hence the two retained coefficients should be 140 and -78.40 . [25%]

(d) **RMS error**

For an orthonormal transform, the total squared error in the reconstructed image will equal the total squared error in the transform coefficients (as explained above). If the two smaller coefficients are set to zero and the two larger coefficients are rounded to values of 150 and -75 by the $25n$ quantiser, then:

Total squared error $= (140 - 150)^2 + (-78.4 + 75)^2 + (60 - 0)^2 + (-32.47 - 0)^2$
$= 100 + 11.56 + 3600 + 1054.3 = 4765.9$

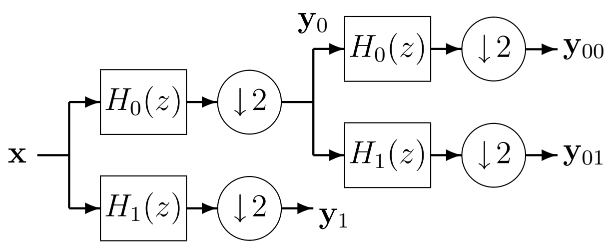
$\therefore$ RMS error over the 16 coefs in the subimage $= \sqrt{4765.9/16} = \sqrt{297.87} = 17.259$

Note that due to orthonormality, it is *not* necessary to calculate the reconstructed subimage to obtain the rms error.

Also note how small the squared error due to the quantiser is (111.56), compared with that due to suppressing the smaller coefficients(4654.3). [25%]

4 (a) **Two-level Wavelet transforms**

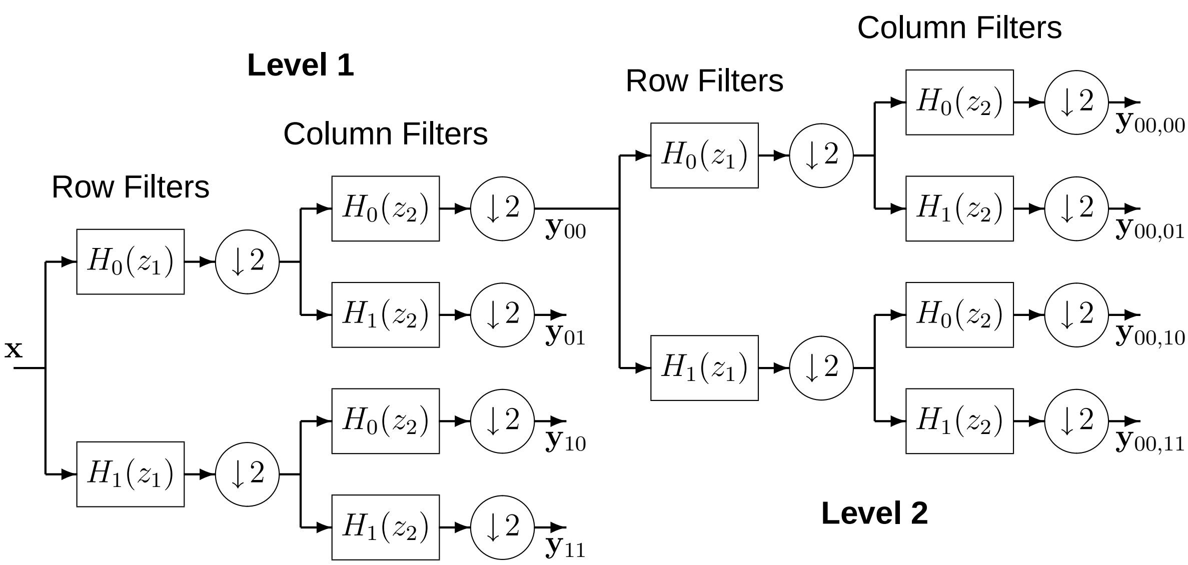
The purpose of the 2-band filter bank in the Fig. 1 of the question is to compress most of the signal energy into the low-frequency band. We may achieve greater compression if the low band is further split into two. This may be repeated a number of times to give a binary filter tree. A version shown with 2 levels in the diagram below.



Two-level Wavelet Analysis tree for 1-D signals

The inverse wavelet transform is achieved by using the inverse filters (from Fig. 1b in the question) in the reverse of the above tree. First we use filters G_0 and G_1 to calculate $\mathbf{y}_0$ from $\mathbf{y}_{00}$ and $\mathbf{y}_{01}$, and then we use them again to calculate $\mathbf{x}$ from $\mathbf{y}_0$ and $\mathbf{y}_1$.

The 2-level tree may be extended for 2-D images, by applying the filters in turn to the rows and the columns of the image, to create four subimages per level. The $\mathbf{y}_{00}$ subimage from level 1, is then split in the same way to create four subimages at level 2. This is shown below.



Two-level Wavelet Analysis tree for 2-D signals [25%]

(b) **Perfect Reconstruction**

Perfect Reconstruction is the condition that, if the inputs of Fig. 1b in the question are connected directly to the outputs of Fig. 1a, then the final output $\hat{X}(z)$ should equal the input $X(z)$ for any input signal. This means that the inverse transform will exactly invert the forward transform process, with zero error.

It is a standard result of multirate filter theory that the process of downsampling by 2, followed by upsampling by 2, shown in Fig. 1, can be represented by

$$\hat{Y}_0(z) = \tfrac{1}{2}[Y_0(z) + Y_0(-z)] \qquad \text{and} \qquad \hat{Y}_1(z) = \tfrac{1}{2}[Y_1(z) + Y_1(-z)]$$

Hence we see that

$$\begin{aligned}
\hat{X}(z) &= G_0(z)\hat{Y}_0(z) + G_1(z)\hat{Y}_1(z) \\
&= \tfrac{1}{2}G_0(z)[Y_0(z) + Y_0(-z)] + \tfrac{1}{2}G_1(z)[Y_1(z) + Y_1(-z)] \\
&= \tfrac{1}{2}G_0(z)H_0(z)X(z) + \tfrac{1}{2}G_0(z)H_0(-z)X(-z) \\
&\quad + \tfrac{1}{2}G_1(z)H_1(z)X(z) + \tfrac{1}{2}G_1(z)H_1(-z)X(-z) \\
&= \tfrac{1}{2}X(z)[G_0(z)H_0(z) + G_1(z)H_1(z)] \\
&\quad + \tfrac{1}{2}X(-z)[G_0(z)H_0(-z) + G_1(z)H_1(-z)]
\end{aligned}$$

If we require $\hat{X}(z) \equiv X(z)$ — *the Perfect Reconstruction (PR) condition* — then:

$$G_0(z)H_0(z) + G_1(z)H_1(z) \equiv 2$$

and

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) \equiv 0$$

Now if $H_1(z) = z^{-1}G_0(-z)$ and $G_1(z) = zH_0(-z)$, as given, then

$$G_0(z)\ H_0(-z) + G_1(z)\ H_1(-z) = G_0(z)\ H_0(-z) + z\,H_0(-z)\ .\ (-z^{-1})\,G_0(z) = 0$$

as required. Hence the result is proved. [25%]

(c) **Filter design**

Substituting for G_1 and H_1 in the first of the above conditions, we see that

$$G_0(z)H_0(z) + G_1(z)H_1(z) = G_0(z)H_0(z) + H_0(-z)G_0(-z) = P_0(z) + P_0(-z) = 2$$

where we have defined a product filter, $P_0(z) = G_0(z)H_0(z)$.

The odd powers of z in P_0 will cancel in the above expression. Hence to make the result always equal to 2, it is only necessary for the coefficients of the even powers of z in P_0 to be zero, apart from the coefficient of z^0 which must be unity.

Applying this to the expressions given for $G_0(z)$ and $H_0(z)$:

$$\begin{aligned}
P_0(z) &= (az + b + cz^{-1})\ .\ (-\tfrac{1}{4}z^2 + \tfrac{1}{2}z + \tfrac{3}{2} + \tfrac{1}{2}z^{-1} - \tfrac{1}{4}z^{-2}) \\
&= -\tfrac{a}{4}z^3 + (\tfrac{a}{2} - \tfrac{b}{4})z^2 + (\tfrac{3a}{2} + \tfrac{b}{2} - \tfrac{c}{4})z + (\tfrac{a}{2} + \tfrac{3b}{2} + \tfrac{c}{2}) \\
&\quad + (-\tfrac{a}{4} + \tfrac{b}{2} + \tfrac{3c}{2})z^{-1} + (-\tfrac{b}{4} + \tfrac{c}{2})z^{-2} - \tfrac{c}{4}z^{-3}
\end{aligned}$$

Hence

$$P_0(z) + P_0(-z) = (a - \tfrac{b}{2})z^2 + (a + 3b + c) + (c - \tfrac{b}{2})z^{-2}$$

For this to equal 2, we require that

$$a - \tfrac{b}{2} = 0 \qquad a + 3b + c = 2 \qquad c - \tfrac{b}{2} = 0$$

and so

$$a = c = \tfrac{b}{2} \qquad \text{and} \qquad 4b = 2$$

Hence

$$b = \tfrac{1}{2} \qquad \text{and} \qquad a = c = \tfrac{1}{4}$$

[25%]

(d) **Image quality**

In a wavelet compression system in which the reconstruction filters are different from the analysis filters, it is found that best reconstructed image quality is achieved if the smoother lowpass filters are used for reconstruction. This is because compression normally removes most of the highpass coefficients at level 1 (and probably at level 2 also), and so the output image is largely composed of weighted lowpass basis functions. Smooth basis functions in general produce better image quality.

The level 2 lowpass basis function is given by $G_0(z)\ G_0(z^2)$.

For the filter we have just designed,

$$\begin{aligned} G_0(z)\ G_0(z^2) &= (az + b + cz^{-1}) \,.\, (az^2 + b + cz^{-2}) \\ &= \tfrac{1}{4}(z + 2 + z^{-1}) \,.\, \tfrac{1}{4}(z^2 + 2 + z^{-2}) \\ &= \tfrac{1}{16}(z^3 + 2z^2 + 3z + 4 + 3z^{-1} + 2z^{-2} + z^{-3}) \end{aligned}$$

This is a smooth triangular function. If the filters were swapped, then the reconstruction basis functions would be based on H_0, and we find that

$$\begin{aligned} H_0(z)\ H_0(z^2) &= (-\tfrac{1}{4}z^2 + \tfrac{1}{2}z + \tfrac{3}{2} + \tfrac{1}{2}z^{-1} - \tfrac{1}{4}z^{-2}) \,.\, (-\tfrac{1}{4}z^4 + \tfrac{1}{2}z^2 + \tfrac{3}{2} + \tfrac{1}{2}z^{-2} - \tfrac{1}{4}z^{-4}) \\ &= \tfrac{1}{16}(z^6 - 2z^5 - 8z^4 + 2z^3 + 7z^2 + 16z + 32 \\ &\quad + 16z^{-1} + 7z^{-2} + 2z^{-3} - 8z^{-4} - 2z^{-5} + z^{-6}) \end{aligned}$$

With its mixture of positive and negative values, this is clearly a less smooth function than G_0 gives, so we conclude that we should *not* swap the filters.

[Calculation of this final convolution is not really necessary to see this result.] [25%]

J Lasenby and N G Kingsbury, Sept 2004.