1 Solution:

a) (i)

$a_{12}^2 - a_{11} \cdot a_{22} = x^2y^2 - x^2y^2 \equiv 0$

The equation is a parabolic equation;

. characteristic equation

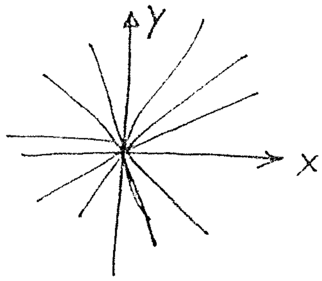
$$x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + y^2 = 0$$

. Solve the characteristic equation:

$$\left(x\frac{dy}{dx} - y\right)^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}, \quad y = cx,$$

characteristics: $y/x = c$

Straight lines from the origin.



[25%]

(ii) . let $\xi = y/x$, $\eta = y$, $\Rightarrow \dfrac{\partial^2 u}{\partial \eta^2} = 0$

. integrate $\dfrac{\partial^2 u}{\partial \eta^2} = 0$, $\dfrac{\partial u}{\partial \eta} = f_1(\xi)$; $u(\xi, \eta) = \eta \cdot f_1(\xi) + f_2(\xi)$

$\underline{u(x,y) = y \cdot f_1(y/x) + f_2(y/x)}$ the general solution [25%]

b) . from B.C.s. guess the solution has form:

$$f_1 = Ae^{-\xi} = Ae^{-\frac{y}{x}}; \quad f_2 = Be^{-\xi^2} = Be^{-\left(\frac{y}{x}\right)^2};$$

$$u = y \cdot f_1 + f_2 = Aye^{-\frac{y}{x}} + Be^{-\left(\frac{y}{x}\right)^2}$$

on $y = 1$, $u = \alpha e^{-\frac{1}{x}} + \beta e^{-\left(\frac{1}{x}\right)^2}$, $\Rightarrow f_1 = \alpha e^{-\frac{y}{x}}$, $f_2 = \beta e^{-\left(\frac{y}{x}\right)^2}$.

$A = \alpha$; $B = \beta$, $u(x,y) = \alpha \cdot y e^{-\left(\frac{y}{x}\right)} + \beta e^{-\left(\frac{y}{x}\right)^2}$

it satisfies the B.C.s on $x = 0$ and $y = 0$

$\therefore$ $u(x,y) = \alpha \cdot y \cdot e^{-\left(\frac{y}{x}\right)} + \beta \cdot e^{-\left(\frac{y}{x}\right)^2}$, $x \in (0, +\infty)$, $y \in [0, 1]$

is the solution sought. [50%]

2. Solution.

a) The PDE for the Green's function and the associated boundary/initial conditions are:

$$\begin{cases} G_{tt} - a^2 G_{xx} = \delta(x - x_0)\,\delta(t - t_0) \\ G(x,0) = 0 \ ; \quad G_t(x,0) = 0 \end{cases}$$

let $\tau = t - t_0$.

$$\begin{cases} G_{\tau\tau} - a^2 G_{xx} = 0 \\ G|_{\tau=0} = 0 \ ; \quad G_\tau|_{\tau=0} = \delta(x - x_0) \end{cases}$$

[20%]

b) from D'Alembert's solution:

$$G(x,t,x_0,t_0) = G(x,\tau,x_0) = \frac{1}{2a}\int_{x-a\tau}^{x+a\tau} \delta(\alpha - x_0)\,d\alpha = \frac{1}{2a}\int_{x-a(t-t_0)}^{x+a(t-t_0)} \delta(\alpha - x_0)\,d\alpha$$

[20%]

c)
$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G\cdot f\,dx_0\,dt_0 = \frac{1}{2a}\int_0^t \int_{x-a(t-t_0)}^{x+a(t-t_0)} \int_{-\infty}^{\infty} \delta(\alpha - x_0)\,f(x_0,t_0)\,dx_0\,d\alpha\,dt_0$$
$$= \frac{1}{2a}\int_0^t \int_{x-a(t-t_0)}^{x+a(t-t_0)} f(\alpha, t_0)\,d\alpha\,dt_0$$

[30%]

d) Uniqueness: let u_1 & u_2 be both the solutions of the problem. the difference $u = u_1 - u_2$ satisfies homogenous equation and boundary/initial conditions:

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

from causality principle $u(x,t) \equiv 0$. thus $u_1 \equiv u_2$.

Stability: let u_1 & u_2 be the solutions corresponding to f_1 and f_2 respectively, the upper bound of the difference satisfy the inequality

$$\|u_1 - u_2\| \le \frac{1}{2a}\int_0^t \int_{x-a(t-t_0)}^{x+a(t-t_0)} \|f_1 - f_2\|\,d\alpha\,dt_0 = \frac{t^2}{2a}\|f_1 - f_2\|,$$

for a limited $t = T$, given an arbitrary small ε, there exists a $\delta = \frac{\varepsilon \cdot 2a}{T^2}$ that if $\|f_1 - f_2\| \le \delta$, $\|u_1 - u_2\| \le \varepsilon$ thus the solution is stable.

[30%]

3 a) $$u = \frac{1}{2}\int_0^L \left\{ EI\left(\frac{\partial\psi}{\partial x}\right)^2 + k\left(\frac{\partial y}{\partial x} - \psi\right)^2 - \omega^2 m y^2 - \omega^2 I_p \psi^2 \right\} dx$$

$$\delta u = \int_0^L \left\{ EI\psi'\delta\psi' + k(y'-\psi)(\delta y' - \delta\psi) - \omega^2 m y \delta y - \omega^2 I_p \psi \delta\psi \right\} dx \quad ; \quad \psi' \equiv \frac{\partial \psi}{\partial x} \text{ etc}$$

integrate by parts $\int_0^L k(y'-\psi)\,\delta y'\,dx$

$$= \left[k(y'-\psi)\delta y \right]_0^L - \int_0^L k(y''-\psi')\delta y\, dx$$

integrate by parts

$$\int_0^L EI\psi'\delta\psi'\,dx$$

$$= \left[EI\psi'\delta\psi \right]_0^L - \int_0^L EI\psi''\delta\psi\, dx$$

$$\delta u = \left[EI\psi'\delta\psi \right]_0^L + \left[k(y'-\psi)\delta y \right]_0^L - \int_0^L \left\{ k(y''-\psi') + \omega^2 m y \right\}\delta y\, dx - \int_0^L \left\{ EI\psi'' + \omega^2 I_p \psi + k(y'-\psi) \right\}\delta\psi\, dx$$

$\delta u = 0$ for all $\delta\psi$ and δy then gives

$$\left.\begin{aligned} &\underline{k(y''-\psi') + \omega^2 m y = 0} \\ &\underline{EI\psi'' + k(y'-\psi) + \omega^2 I_p \psi = 0} \end{aligned}\right\} \text{required differential equations}$$

On the boundaries

$$\left.\begin{aligned} EI\psi'\delta\psi = 0 &\Rightarrow \underline{EI\psi' = 0 \text{ or } \psi = 0} \\ k(y'-\psi)\delta y = 0 &\Rightarrow \underline{k(y'-\psi) = 0 \text{ or } y = 0} \end{aligned}\right\} \text{required b.c.'s}$$

[40 %]

b) $$k(y''-\psi') + \omega^2 m y = 0 \quad \Rightarrow \quad y'' - \psi' + \frac{\omega^2 m y}{k} = 0$$

$$\Rightarrow \quad EI y^{IV} - EI\psi''' + (\omega^2 m/k) y''$$

$$EI\psi'' + k(y'-\psi) + \omega^2 I_p \psi = 0 \quad \Rightarrow \quad -EI\psi''' = k(y''-\psi') + \omega^2 I_p \psi'$$

$$= ky'' + (-k + \omega^2 I_p)\psi'$$

$$= ky'' + (-k + \omega^2 I_p)\left[y'' + \frac{\omega^2 m}{k} y \right]$$

$$= \omega^2 I_p y'' + (-k + \omega^2 I_p)(\omega^2 m/k) y$$

$$\Rightarrow \quad EI y^{IV} + \omega^2 I_p y'' + \omega^2 m\left(-1 + \frac{\omega^2 I_p}{k}\right) y + (\omega^2 m/k) y'' EI = 0$$

$$\Rightarrow \quad \underline{EI y^{IV} + \omega^2 (I_p + EIm/k) y'' - \omega^2 m (1 - \omega^2 I_p/k) y = 0}$$

[30 %]

c) The variation $\delta V = 0$ would need to produce:

$$\int_0^L \left\{ EIy^{iv} + \omega^2 (I_p + EIm/k) y'' - \omega^2 m (1 - \omega^2 I_p/k) y \right\} \delta y \, dx = 0$$

$$\Rightarrow \quad \underline{V = \tfrac{1}{2} \int_0^L \left\{ EIy''^2 - \omega^2 (I_p + EIm/k) y'^2 - \omega^2 m (1 - \omega^2 I_p/k) y^2 \right\} dx} \qquad [30\,\%]$$

4 a)

$$\int_S \nabla\phi\,(\nabla\phi.\underset{\sim}{n})\,ds = \int_S \phi_i\,\phi_j n_j\,ds \quad \text{where } \phi_i \equiv \partial\phi/\partial x_i \text{ etc}$$

$$= \int_V \frac{\partial}{\partial x_j}(\phi_i\phi_j)\,dV = \int_V (\phi_{ij}\phi_j + \underbrace{\phi_i\phi_{jj}}_{\text{This is } \nabla^2\phi\text{, hence }=0})\,dV$$

$$\int_S (\nabla\phi.\nabla\phi)\,\underset{\sim}{n}\,ds = \int_S \phi_j\phi_j n_i\,ds$$

$$= \int_V \frac{\partial}{\partial x_i}(\phi_j\phi_j)\,dV = \int_V \phi_{ji}\phi_j \times 2\,dV = 2\int_V \phi_{ij}\phi_j\,dV$$

Thus $\underline{\int_S \nabla\phi\,(\nabla\phi.\underset{\sim}{n})\,ds = \frac{1}{2}\int_S (\nabla\phi.\nabla\phi)\,\underset{\sim}{n}\,ds}$ [30 %]

b) (i)

$$\int_S p\,\underset{\sim}{n}\,ds = -\frac{D}{Dt}\int_V \rho\,\underset{\sim}{u}\,dV$$

$$= -\int_V \rho\,\frac{\partial \underset{\sim}{u}}{\partial t}\,dV - \int_S \rho\,\underset{\sim}{u}\,(\underset{\sim}{u}.\underset{\sim}{n})\,ds$$

$$= -\int_V \rho\,\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial t}\right)dV - \int_S \rho\,\nabla\phi\,(\nabla\phi.\underset{\sim}{n})\,ds$$

From part (a)

$$= -\int_S \rho\,\frac{\partial\phi}{\partial t}\,\underset{\sim}{n}\,ds - \int_S \rho(\nabla\phi.\nabla\phi)\,\underset{\sim}{n}\,ds\,\tfrac{1}{2}$$

Valid for all surfaces S $\Rightarrow$ $\underline{p = -\rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho\,\underset{\sim}{u}.\underset{\sim}{u}}$ [20 %]

(ii) Take the grad of the previous result : $p = -\rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho\,\phi_j\phi_j$

$$\Rightarrow p_i = -\rho\frac{\partial\phi_i}{\partial t} - \frac{1}{2}\rho\,(\phi_j\phi_j)_i$$

$$\Rightarrow p_i = -\rho\frac{\partial\phi_i}{\partial t} - \rho\,\underbrace{\phi_{ji}\phi_j}_{\frac{\partial u_i}{\partial x_j}u_j = (\underset{\sim}{u}.\nabla\underset{\sim}{u})_i}$$

$$\Rightarrow \underline{\nabla p = -\rho\frac{\partial\underset{\sim}{u}}{\partial t} - \rho(\underset{\sim}{u}.\nabla\underset{\sim}{u})}$$ [20 %]

(iii) $\frac{D}{Dt}\int_V \rho\, dV = \int_V \underbrace{\frac{\partial\rho}{\partial t}}_{0} dV + \int_S \rho\, \underset{\sim}{u}.\underset{\sim}{n}\, ds = \int_S \rho u_i n_i\, ds = \int_V \rho u_{i,i}\, dV$

$= \int_V \rho\, \phi_{ii}\, dV = \int_V \rho \nabla^2\phi\, dV$

$\underline{= 0}$ [10 %]

c) $p = -\rho \frac{\partial\phi}{\partial t} - \frac{1}{2}\rho\, \phi_i \phi_i$

$\Rightarrow p_j = -\rho \frac{\partial\phi_j}{\partial t} - \frac{1}{2}\rho\, (\phi_i\phi_i)_j = -\rho\frac{\partial\phi_j}{\partial t} - \rho\, \phi_i\phi_{ij}$

$\Rightarrow p_{jj} = -\rho \underbrace{\frac{\partial\phi_{jj}}{\partial t}}_{\frac{\partial}{\partial t}(\nabla^2\phi)=0} - \rho\,(\phi_i\phi_{ij})_j = -\rho\,(\phi_{ij}\phi_{ij}) - \rho\,\underbrace{(\phi_i\phi_{ijj})}_{\frac{\partial}{\partial x_i}(\nabla^2\phi)=0}$

$\Rightarrow \nabla^2 p = -\rho\, \phi_{ij}\phi_{ij} = \underline{-\rho \sum_i\sum_j \left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)^2}$ [20 %]