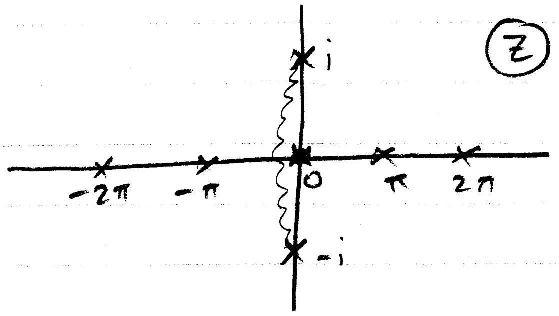
CRIB FOR Module 4M13 2003/4

1. (a) $f(z) = \dfrac{1}{\sqrt{z+i}\,\sqrt{z-i}\,\sin z}$ 15%

has branch cuts at $z = \pm i$ and simple poles at $z = 0, \pm\pi, \pm 2\pi \ldots$



1. (b) $f(z) = \dfrac{e^{i4z}}{z - \sin z}$

40%

Recall that $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} \cdots$

$$\Rightarrow \quad z - \sin z = \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots$$

This has a zero of order 3 at $z=0$.
Hence, $f(z)$ has a pole of order 3 at $z=0$.

$$\frac{1}{z - \sin z} \approx \frac{1}{\frac{z^3}{3!}\left(1 - \frac{3!}{5!}z^2 + \frac{3!}{7!}z^4 \cdots\right)}$$

$$= \frac{3!}{z^3}\left(1 + \frac{3!}{5!}z^2 - \frac{3!}{7!}z^4 + \cdots\right) \quad \text{for small } z$$

Now expand the top line.

$$e^{i4z} = 1 + i4z + \frac{(i4z)^2}{2!} + \frac{(i4z)^3}{3!} + \cdots$$

$$= 1 + i4z - 8z^2 - i\frac{32}{3}z^3 + \cdots$$

The Laurent Series expansion is of the form,

$$f(z) \approx \underbrace{\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z}}_{\text{expand up to these terms}} + a_0 + a_1 z + a_2 z^2 + \cdots$$

Here,

$$f(z) = \frac{3!}{z^3}\left(1 + \frac{3!}{5!}z^2 + \cdots\right)\left(1 + i4z - 8z^2 - i\frac{32}{3}z^3 + \cdots\right)$$

$$= \frac{3!}{z^3}\left(1 + i4z + \left(\frac{3!}{5!} - 8\right)z^2 + z^3\left(-i\frac{32}{3} + \frac{4!}{5!}i\right) + \cdots\right)$$

So, $a_{-3} = 3!$ $\quad a_{-2} = i4!$

Residue $= a_{-1} = 3!\left(\frac{3!}{5!} - 8\right)$ $\qquad a_0 = i\left(\frac{4!}{5!} - \frac{32}{3}\right)$

45%

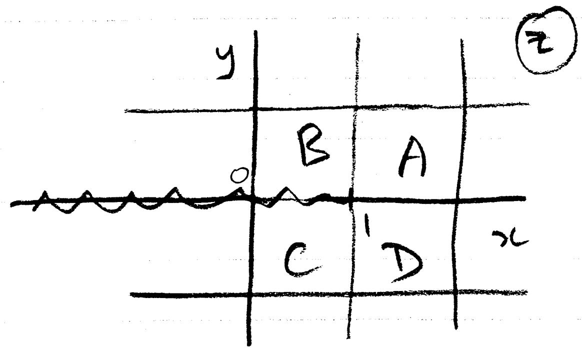
1 (c) (i) $w(z) = \sqrt{z-1} + 1$

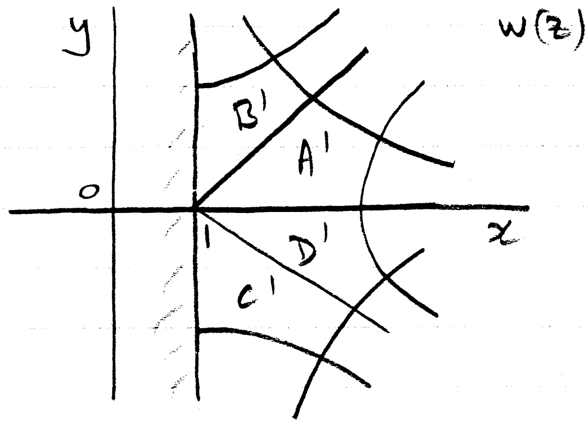
Branch pts. at $z = 1, \infty$.
Critical pt. at $w'(z) = 0, \infty$

$$w'(z) = \frac{1}{2\sqrt{z-1}}$$

$w' = 0$ at $z = \infty$
$w' = \infty$ at $z = 1$

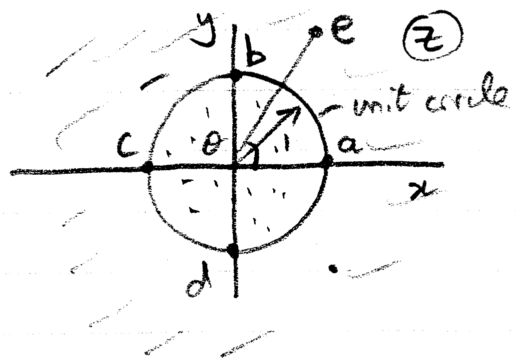
So the critical pts. are at $z = 1, \infty$.



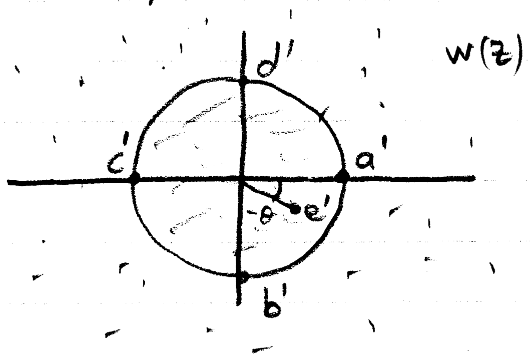


(ii) $w(z) = 1/z$

$w'(z) = \frac{-1}{z^2}$ Critical pts. at $w'(z) = 0, \infty$
$\Rightarrow z = 0, \infty$



$\xrightarrow{w(z)}$



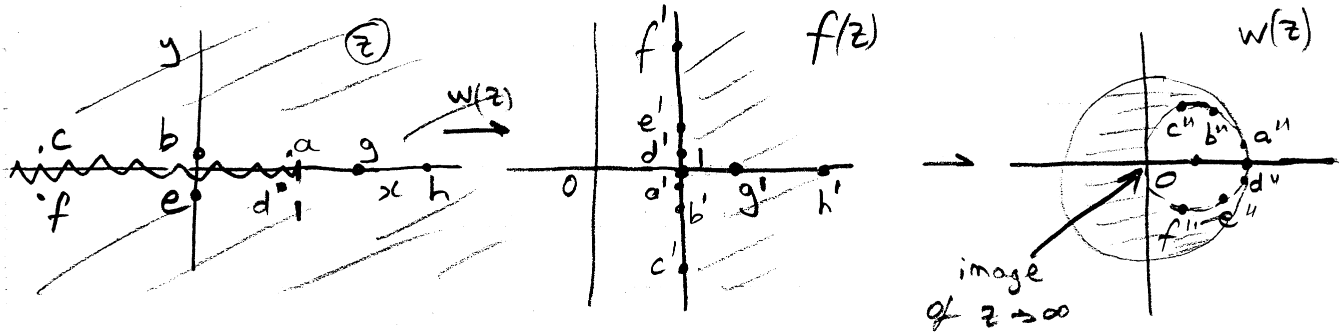
$z = r e^{i\theta}$

$w(z) = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$

1 (c) (iii) $\quad w(z) = \dfrac{1}{\sqrt{z-1}+1}$

Break down into 2 parts :

$$f(z) = \sqrt{z-1} + 1 \quad \text{and} \quad w(z) = \frac{1}{f(z)}$$



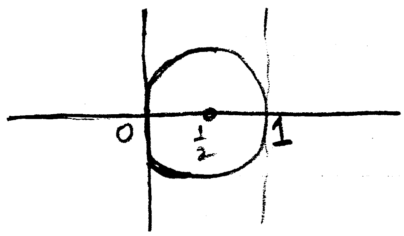
consider the line in the $f(z)$ plane $f(z) = 1 + i\delta$

$$\Rightarrow \quad w = \frac{1}{f(z)} = \frac{1}{1+i\delta} = \frac{1-i\delta}{1+\delta^2}$$

write $\quad w = u + iv \quad \Rightarrow \quad u = \dfrac{1}{1+\delta^2} \quad v = \dfrac{-\delta}{1+\delta^2}$

Note : $\quad u^2 + v^2 = \dfrac{1}{1+\delta^2} = u$

$$\Rightarrow \quad u^2 - u + v^2 = 0 \quad \Rightarrow \quad \left(u - \tfrac{1}{2}\right)^2 + v^2 = \tfrac{1}{4}$$

$\Rightarrow$ 

– locus of a circle in the $w(z)$ plane, of radius $\frac{1}{2}$, and centred at $(\frac{1}{2}, 0)$

2 (a) Consider $I = \int_{-\infty}^{\infty} \frac{1}{4z^2+2z+1}\,dz$ (50%)

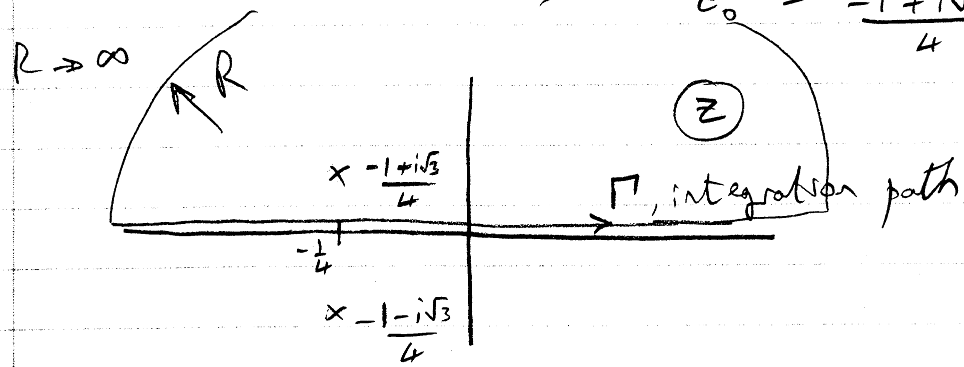
Now $4z^2 + 2z + 1 = 4\left(z^2 + \frac{1}{2} + \frac{1}{4}\right)$

$= 4\left[\left(z+\frac{1}{4}\right)^2 + \frac{3}{16}\right]$

Note roots are at $z_0 = -\frac{1}{4} \pm i\sqrt{\frac{3}{16}}$

$= -\frac{1}{4} \pm i\frac{\sqrt{3}}{4}$

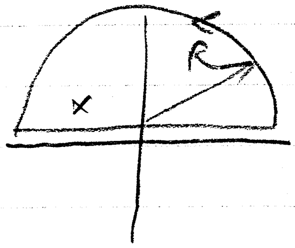
$\Rightarrow \quad z_0 = \frac{-1+i\sqrt{3}}{4}, \quad \frac{-1-i\sqrt{3}}{4}$



So $f(z) = \frac{1}{4(z^2+2z+1)} = \frac{1}{4}\,\frac{1}{(z-z_1)(z-z_2)}$

where $z_1 = \frac{-1+i\sqrt{3}}{4}$ $\quad z_2 = \frac{-1-i\sqrt{3}}{4}$

choose arbitrarily to close the contour in the upper half-plane.



As $R \to 0$ contrib^n to integral $\to 0$

So $I = \oint f(z)\,dz$

$= 2\pi i \times$ sum of residues

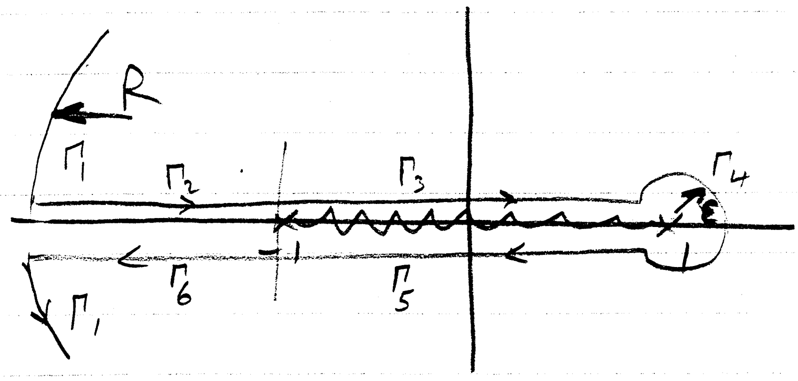
2. (a) Residue at $z = z_1$ is $\dfrac{1}{4(z_1 - z_2)} = \dfrac{1}{i\,2\sqrt{3}}$

$$\Rightarrow \quad I = \frac{2\pi i}{i\,2\sqrt{3}} = \underline{\frac{\pi}{\sqrt{3}}}$$

2. (b) Consider $I \equiv \int_{-1}^{1} \frac{1}{\sqrt{1-z^2}\,(1+z^2)}\, dz$ (50%)

Consider $f(z) = \frac{1}{\sqrt{1+z}\,\sqrt{1-z}\,(z^2+1)}$

This has simple poles at $z = \pm i$, and branch cuts at $z = \pm 1$. Add branch cuts as shown.



Consider $J = \oint f(z)\, dz$

Now $\int_{\Gamma_1} f(z)\, dz \to 0$ as $R \to \infty$ $\quad \int_{\Gamma_4} f(z)\, dz \to 0$ as $\varepsilon \to 0$

$$\int_{\Gamma_2 + \Gamma_6} f(z)\, dz = 0$$

Consider $\int_{\Gamma_3} f(z)\, dz$.

On Γ_3, $f(z) = \frac{1}{\sqrt{1-x^2}\,(1+x^2)} \Rightarrow \int_{\Gamma_3} f(z)\, dz = I$

On Γ_5, $f(z) = \frac{-1}{\sqrt{1-x^2}\,(1+x^2)} \Rightarrow \int_{\Gamma_5} f(z)\, dz = I$

2(b)(ii) contd.

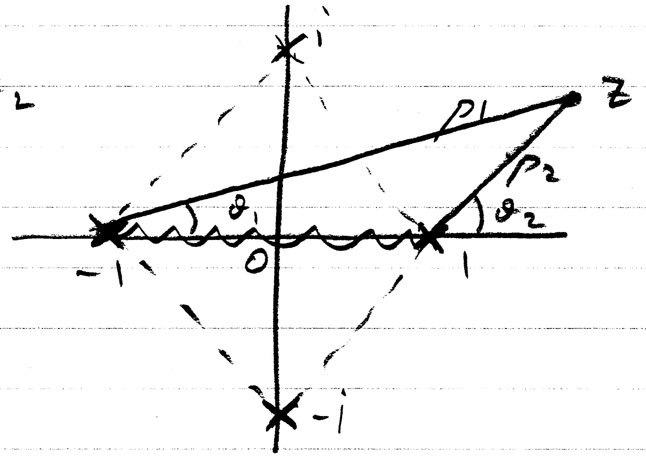
Hence, $J = 2I$ and $J = 2\pi i \times$ (Sum of residues)

The function $f(z) = \dfrac{1}{\sqrt{1+z}\,\sqrt{1-z}\,(z+i)(z-i)}$ has simple poles at $z = \pm i$.

Write $z+1 = \rho_1 e^{i\theta_1}$

$z-1 = \rho_2 e^{i\theta_2}$

$\Rightarrow \quad 1-z = \rho_2 e^{i(\theta_2+\pi)}$



Residue of $f(z)$ at $z = i$?

At $z = i$, $\rho_1 = \sqrt{2}$ $\theta_1 = \pi/4$

$\Rightarrow \quad z+1 = \sqrt{2}\, e^{i\pi/4} \Rightarrow (z+1)^{1/2} = 2^{1/4} e^{i\pi/8}$

$z-1 = ?$ $\rho_2 = \sqrt{2}$ $\theta_2 = e^{i3\pi/4}$

$\Rightarrow \quad z-1 = \sqrt{2}\, e^{i3\pi/4} \Rightarrow (1-z) = \sqrt{2}\, e^{i7\pi/4}$

$\Rightarrow \quad (1-z)^{\frac{1}{2}} = 2^{1/4} e^{i7\pi/8}$

Residue of $f(z)$ at $z = i$ is :

$$\frac{1}{2^{1/4} e^{i\pi/8} \cdot 2^{1/4} e^{i7\pi/8} \cdot 2i} = \frac{i}{2\sqrt{2}}$$

Now calculate the residue of $f(z)$ at $z = -i$.

$z+1 = \rho_1 e^{i\theta_1}$ $\rho_1 = \sqrt{2}$ $\theta_1 = -\pi/4$

$z-1 = \rho_2 e^{i\theta_2}$ $\rho_2 = \sqrt{2}$ $\theta_2 = -3\pi/4$

$\Rightarrow$ Residue of $f(z)$ is $\dfrac{1}{2^{1/4} e^{-i\pi/8}\, 2^{1/4} e^{i\pi/8}\, -2i} = \dfrac{i}{2\sqrt{2}}$

Hence $I = \frac{1}{2} J = \pi i \times \left(\begin{array}{c}\text{sum of}\\ \text{residues}\end{array}\right) = -\pi/\sqrt{2}$

3. (a) Minimise $f(x,y) = -x\left(\frac{y - \mu x}{xy + \mu}\right)$ 15%

$-$ for minimisation

$\lambda > 0 \Rightarrow \tan\lambda = \underline{\underline{x > 0}}$ (1)

as $\tan\lambda$ increases for increasing λ

$\alpha \geqslant 15° \Rightarrow \cos\alpha = \underline{\underline{y \leqslant \cos 15° = 0.966}}$ (2)

as $\cos\alpha$ decreases for increasing α

(b) $\frac{\partial f}{\partial x} = \frac{(-y + 2\mu x)(xy + \mu) - (-xy + \mu x^2)y}{(xy+\mu)^2}$ 20%

$= \frac{\cancel{-xy^2} + 2\mu x^2 y - \mu y + 2\mu^2 x + \cancel{xy^2} - \mu x^2 y}{(xy+\mu)^2}$

$= \frac{\mu(x^2 y + 2\mu x - y)}{(xy+\mu)^2}$

$\frac{\partial f}{\partial y} = \frac{-x(xy+\mu) - (-xy + \mu x^2)x}{(xy+\mu)^2}$

$= \frac{\cancel{-x^2y} - \mu x + \cancel{x^2y} - \mu x^3}{(xy+\mu)^2}$

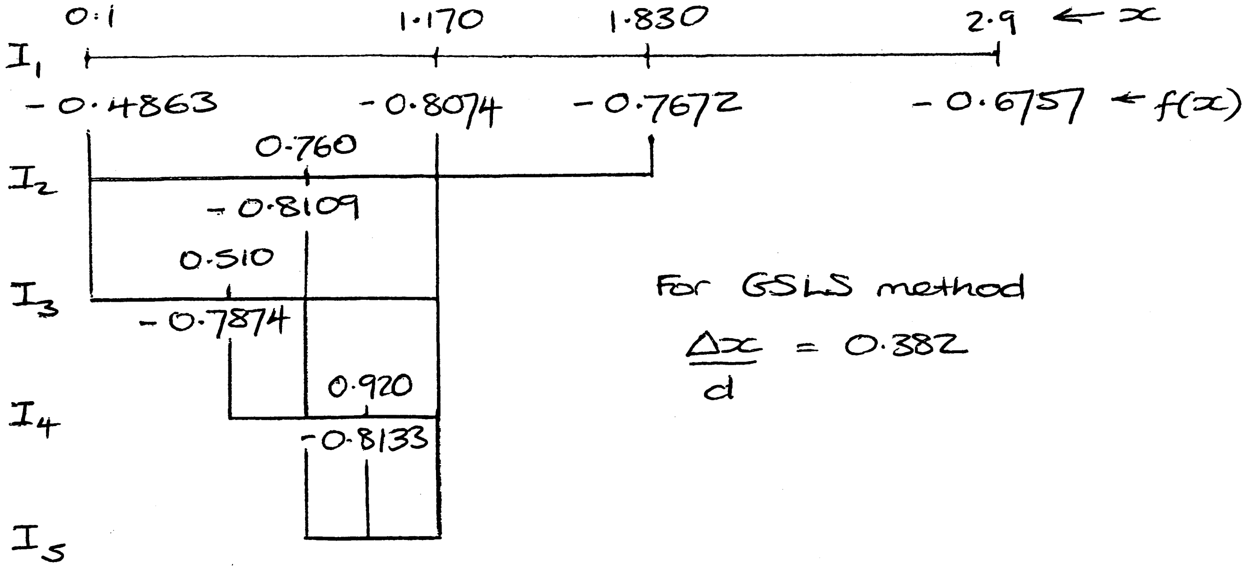
$= -\frac{\mu x(1 + x^2)}{(xy+\mu)^2}$

Hence $\partial f/\partial y$ can only be 0 when $x = 0$, which violates (1), so there can be no stationary point (where $\nabla f = 0$) and thus no unconstrained optimum.

Clearly $\partial f/\partial y$ is $-$ve for all feasible x and y. If $\partial f/\partial y$ is $-$ve then increasing y decreases f so y should be as large as possible, i.e. $y = 0.966$.

3. (c) For $\mu = 0.1$ and $y = 0.966$ 40%

$$f(x) = -\left(\frac{0.966x - 0.1x^2}{0.966x + 0.1}\right)$$



For GSLS method

$$\frac{\Delta x}{d} = 0.382$$

$$\underline{\underline{I_5 = 0.760 \leqslant x \leqslant 1.170}} \qquad (37.2° \leqslant \lambda \leqslant 49.5°)$$

(d) $$\frac{df}{dx} = \frac{\mu(x^2y + 2\mu x - y)}{(xy + \mu)^2}$$ using (b) 25%

$\therefore \dfrac{df}{dx} = 0$ when $x^2y + 2\mu x - y = 0$

$$\therefore \quad x^* = \frac{-2\mu \pm \sqrt{4\mu^2 + 4y^2}}{2y}$$

$x > 0$ $\therefore$ want +ve root

$$\therefore \quad x^* = \frac{-\mu + \sqrt{\mu^2 + y^2}}{y} \qquad (3)$$

For $\mu = 0.1$ and $y = 0.966$ $\Rightarrow$ $\underline{\underline{x^* = 0.902}}$

$(\lambda = 42.1°)$

This confirms that the GSLS is converging on the optimum.

3. (d) continued.

Using (3) $\dfrac{dx^*}{d\mu} = \dfrac{-1 + \mu(\mu^2 + y^2)^{-1/2}}{y}$

For $\mu = 0.1$, $y = 0.966$ $\dfrac{dx^*}{d\mu} = -0.929$

$\therefore$ as μ increases x^* decreases

4. (a) If the constraint on R is not active then it is possible to increase one (or more) failure rate x_i and this will in turn decrease C, thus improving the objective. Thus, the constraint on R must be active at the optimum – otherwise the objective can still be improved. 10%

(b) The problem is 45%

Minimise $C(\underline{x}) = \sum_{i=1}^{n} \frac{c_i}{x_i^2}$

subject to $h(\underline{x}) = \exp\left(-T\sum_{i=1}^{n} x_i\right) - R_o = 0$

Using hint, $h(\underline{x})$ can be formulated as

subject to $h(\underline{x}) = \sum_{i=1}^{n} x_i + \frac{\ln R_o}{T} = 0$

So, using Lagrange multiplier method, optimum occurs where $\nabla C + \lambda \nabla h = 0$

$$\therefore \left[-\frac{2c_1}{x_1^3}, -\frac{2c_2}{x_2^3}, \ldots\right]^T + \lambda\left[1, 1, \ldots\right] = 0$$

$$\therefore \frac{2c_i}{x_i^3} = \lambda \quad \forall\, i$$

with $\sum_{i=1}^{n} x_i = -\frac{\ln R_o}{T}$

$$\therefore \sum_{i=1}^{n} \sqrt[3]{\frac{2c_i}{\lambda}} = -\frac{\ln R_o}{T}$$

$$\therefore \lambda = -\frac{T^3}{(\ln R_o)^3}\left(\sum_{i=1}^{n} \sqrt[3]{2c_i}\right)^3$$

$$\therefore x_i = \sqrt[3]{\frac{2c_i}{\lambda}}$$

$$= -\frac{\sqrt[3]{2c_i}\,\ln R_o}{T\sum_{j=1}^{n}\sqrt[3]{2c_j}} = -\frac{\sqrt[3]{c_i}\,\ln R_o}{T\sum_{j=1}^{n}\sqrt[3]{c_j}}$$

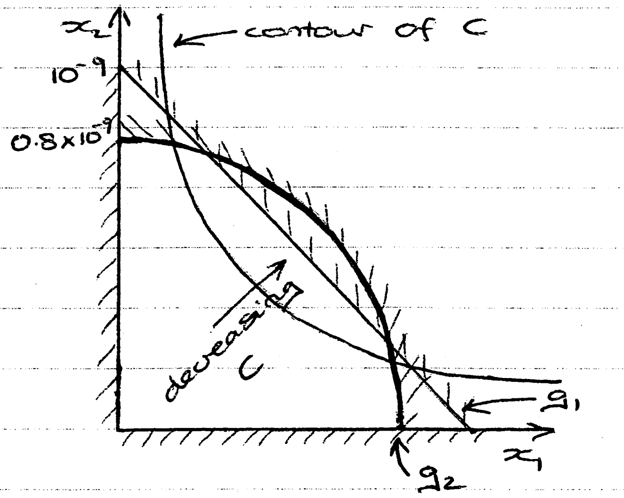
(c) The problem is now 45%.

Minimise $C = \dfrac{1 \times 10^{-13}}{x_1^2} + \dfrac{2 \times 10^{-13}}{x_2^2}$

g_1 subject to $x_1 + x_2 = \dfrac{-\ln R_0}{T} = -\dfrac{\ln 0.9999}{10^5} = 10^{-9}$

g_2 and $6.25 \times 10^{18}(x_1^2 + x_2^2) = 4$

The feasible region is



Contours of C are as shown.
It is clear that at the optimum g_1 will be active but not g_2.
Thus the result for (b) can be used.

$$x_1 = -\frac{\sqrt[3]{1\times10^{-13}}\ \ln 0.9999}{10^5\left\{\sqrt[3]{1\times10^{-13}} + \sqrt[3]{2\times10^{-13}}\right\}} = \underline{\underline{4.425\times10^{-10}\ \text{hr}^{-1}}}$$

Similarly $x_2 = \sqrt[3]{2}\, x_1 = \underline{\underline{5.575\times10^{-10}\ \text{hr}^{-1}}}$

and $\underline{\underline{C = 1.154\times10^{6}\ £}}$