UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Wednesday 28 April 2004 2.30 to 4

Module 4A10

FLOW INSTABILITY

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A10 Data sheet (2 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 A traffic calming scheme is modelled locally (i.e. neglecting end effects) by a thin membrane of mass m / unit area and negligible tension over a layer of inviscid liquid of density ρ and thickness h.

A distant car causes the fluid to flow with velocity U, as illustrated in Fig. 1.

(a) Show that linear perturbations to the surface of the form $z = h + \eta_0 e^{st+ikx}$ lead to the velocity potential of the form

$$\phi(x,z,t) = Ux + Ae^{st+ikx}\cosh(kz)$$

where A is a complex constant. [25%]

(b) Show that disturbances with wavenumber k are only unstable if

$$U^2 > \frac{g}{k}\left(\frac{\rho}{mk} + \tanh(kh)\right)$$ [75%]

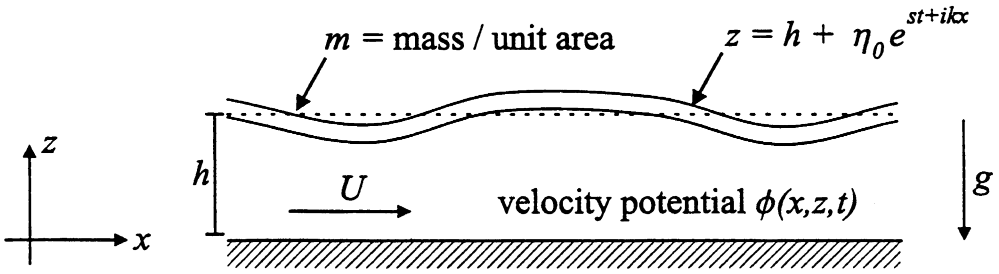

Fig. 1

2 Liquid of density ρ, kinematic viscosity ν, thermal diffusivity κ and coefficient of expansion α, is contained between two horizontal rigid plates a distance d apart, as shown in Fig. 2. The lower plate is at temperature T_0 and the upper at T_1 ($T_0 > T_1$). The equilibrium pressure on the lower plate is $\overline{p_0}$.

(a) The equations of motion using the Boussinesq approximation are stated on the data sheet. What physical assumptions are made in the Boussinesq approximation? [15%]

Assume this approximation for the configuration in Fig. 2.

(b) What is the temperature and pressure variation in the equilibrium configuration in Fig. 2? [15%]

(c) Explain physically why you would expect this configuration to be unstable. [10%]

(d) Introducing suitable non-dimensional parameters, derive the non-dimensional form of the equations of motion for linearised perturbations from the equilibrium configuration. State clearly the variables used to non-dimensionalise time, velocity, pressure and temperature. [40%]

(e) Explain the physical meaning of the Rayleigh number and why it describes the susceptibility to instability. [10%]

(f) Why does the first critical flow condition depend only on the Rayleigh number and not on the Prandtl number ν / κ ? [10%]

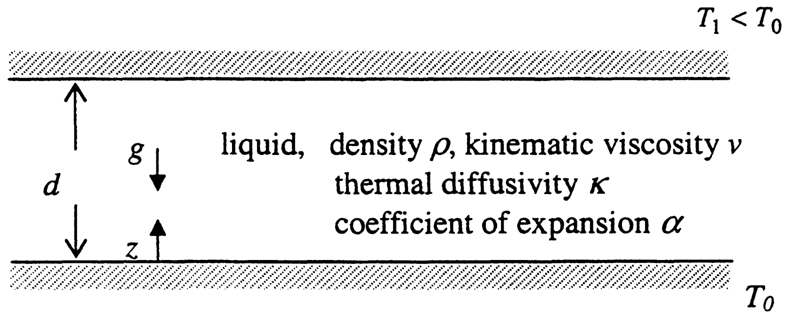

Fig. 2

3 Figure 3 shows a heat exchanger consisting of two long tubes of radius R_1 and R_2 both filled with the same fluid. When unperturbed, the tubes are concentric. The tubes are constrained to move in the horizontal direction and can vibrate very small distances around the centre-line.

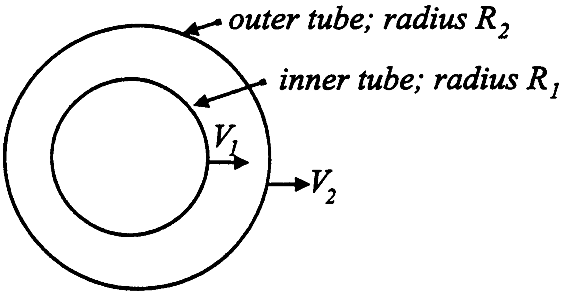

Fig. 3

The instantaneous velocities of the cylinders are V_1 and V_2. The velocity potential of the fluid between the tubes is given by:

$$\phi(r,\theta,t) = a_1 r\cos\theta + \frac{a_2}{r}\cos\theta$$

where r is defined from the centreline of the unperturbed cylinders and θ is measured from the horizontal. In addition,

$$a_1 = \frac{V_2 R_2^2 - V_1 R_1^2}{R_2^2 - R_1^2}$$

and

$$a_2 = \frac{(V_2 - V_1) R_1^2 R_2^2}{R_2^2 - R_1^2}$$

(a) Using the chain rule, derive an expression for $\partial\phi/\partial t$ in terms of dV_1/dt dV_2/dt, R_1, R_2, r and θ. [10%]

(b) In this linearised problem, the unsteady form of the Bernoulli equation is:

$$p = p_\infty - \rho\left(\frac{\partial\phi}{\partial t}\right)$$

(Cont.

Show that the horizontal force on the central cylinder due to the fluid is:

$$F_1 = -M_1 \frac{dV_1}{dt} + M_{12} \frac{dV_2}{dt}$$

where $$M_1 = \rho \pi R_1^2 \left(\frac{(R_2/R_1)^2 + 1}{(R_2/R_1)^2 - 1} \right)$$

and $$M_{12} = \rho \pi R_2^2 \left(\frac{2}{(R_2/R_1)^2 - 1} \right)$$ [40%]

(c) What does M_1 represent physically? Sketch the variation of M_1 with R_2/R_1. [10%]

(d) The central tube is very thin and the mass of its walls can be neglected relative to the mass of fluid it contains. When the space between the tubes is empty but the central tube is full, the central tube has a natural frequency of 100Hz. Estimate its natural frequency when $R_2/R_1 = 2$ and both tubes are full. [20%]

(e) An experiment is devised to measure M_1 as a function of R_2/R_1 for fluids of different viscosities. Show the plots of M_1 against R_2/R_1 that you would expect to see as the viscosity increases. How will the natural frequency of the central tube vary as the viscosity increases? What other added coefficients would play a rôle? [10%]

(f) What does M_{12} represent physically? [10%]

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4 The owners of London's newest skyscraper are expanding to the site next door. They commission a famous architect to develop a bold new image. The architect proposes two towers linked by a square-sectional walkway (Fig. 4).

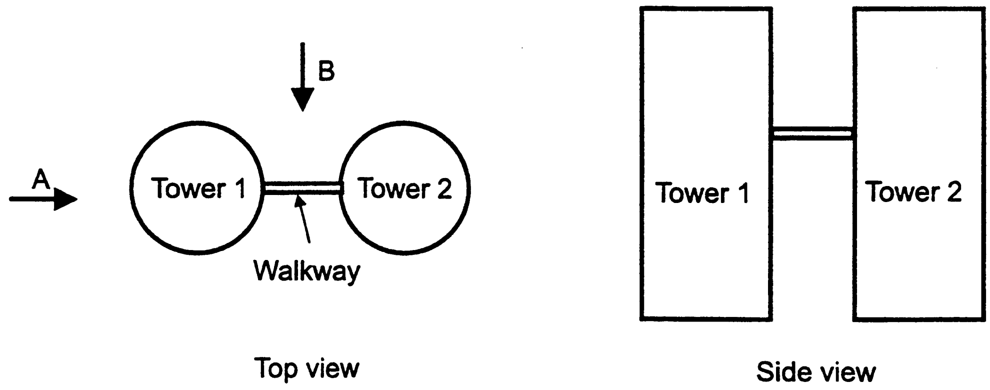

Fig. 4

(a) You have been employed, as a flow-structure interaction expert, to review the design. What motion would you expect to see in the towers and walkway when the wind comes from

(i) direction A and
(ii) direction B?

How and why would this motion arise? What measures could you propose to prevent it? [50%]

(b) The architect presses on regardless and two years later the building is complete. If one assumes that motion of the walkway can be represented by a simple harmonic oscillator, the equation of motion is:

$$m\ddot{y} + 2m\zeta\omega_n\dot{y} + ky = F(t)$$

where $$F(t) = \tfrac{1}{2}\rho U^2 D C_y .$$

(Cont.

m = mass per unit length of walkway
ζ = damping factor
ω_n = natural frequency of walkway
k = stiffness
U = wind velocity
ρ = density of air
D = height of walkway (distance from bottom to top, *not* height above ground)
C_y = aerodynamic force coefficient
α = apparent angle of attack

Show that the walkway will begin to gallop from soft excitation when:

$$\frac{1}{2}\rho U D \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0} > 2m\zeta\omega_n$$ [30%]

c) The design is such that $D = 3$ m, $m = 1000$ kg m^{-1}, $\zeta = 0.1$, $\omega_n = 0.5$ rad s^{-1}, $\rho = 1.3$ kg m^{-3} . C_y for a square cross-section is given in Fig. 5. At what windspeed will the walkway start to gallop? [10%]

d) At what frequency would you expect this to happen? [10%]

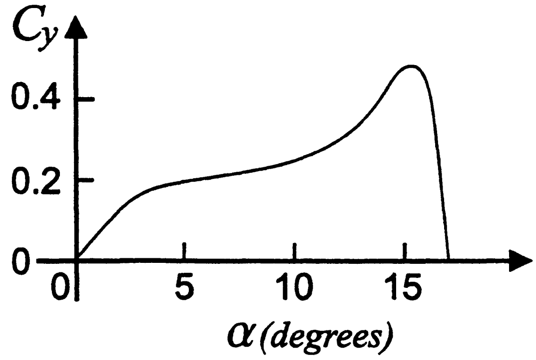

Fig. 5

END OF PAPER

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EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla \phi \text{ and } \nabla^2 \phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u,v,w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x,\theta,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma \left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 \eta}{\partial \theta^2} \right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is unstable / stable to inviscid axisymmetric disturbances if Γ^2 decreases / increases with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu \left(\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} \right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2} = 0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + (1 - \alpha(T - T_0))\boldsymbol{g} + \upsilon\nabla^2\boldsymbol{u}$$

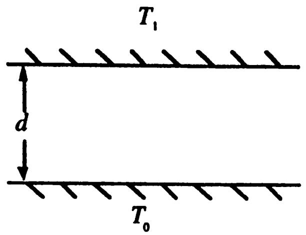
and

$$\frac{DT}{Dt} = \kappa\nabla^2 T$$

Rayleigh-Bénard convection

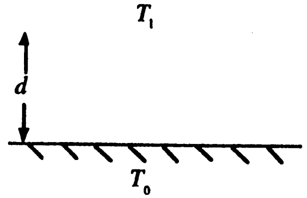
A fluid between two **rigid** plates is unstable when

$$\text{Ra} \geq 1708$$



A liquid with a **free** upper surface is unstable when

$$\frac{\text{Ra}}{\text{Ra}_c} + \frac{\text{Ma}}{\text{Ma}_c} \geq 1$$



where

$$\text{Ra} = \frac{g\alpha(T_0 - T_1)d^3}{\upsilon\kappa}, \quad \text{Ma} = \frac{\chi(T_0 - T_1)d}{\rho\upsilon\kappa} \quad \text{with } \chi = -\frac{d\sigma}{dT}$$

$\text{Ra}_c \approx 670$ $\quad$ $\text{Ma}_c \approx 80.$

USEFUL MATHEMATICAL FORMULA

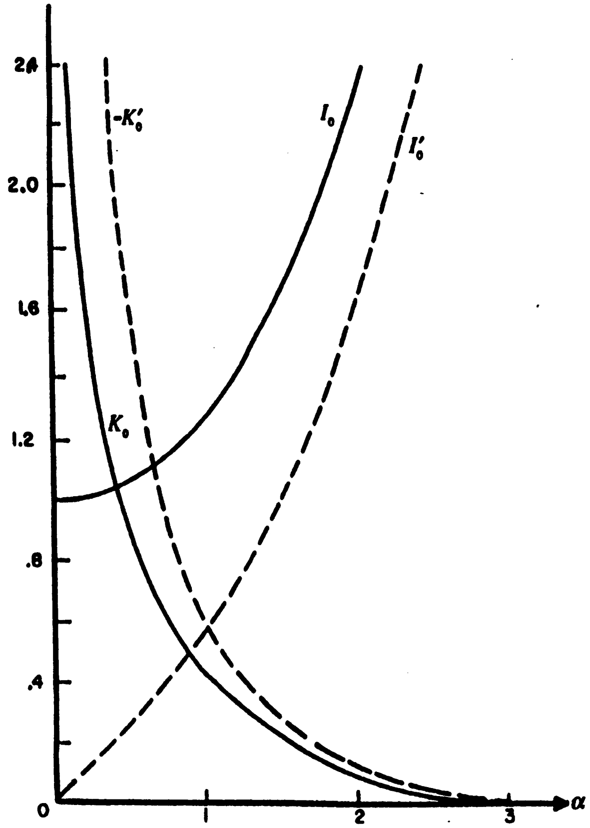
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - k^2 f = 0.$$

$I_0(kr)$ is finite at $r = 0$ and tends to infinity as $r \to \infty$,

$K_0(kr)$ is infinite at $r = 0$ and tends to zero as $r \to \infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative