
Friday 23 April 2004

9 to 10.30

Module 4C1

DESIGN AGAINST FAILURE

Answer not more than three questions.

All questions carry the same number of marks.

The approximate percentage of marks allocated to each part of a question is indicated in the right margin.

Attachments:

- (i) Fracture mechanics datasheet (8 pages)*
- (ii) Elasticity and plasticity formulae (2 pages)*

**You may not start to read the questions
printed on the subsequent pages of this
question paper until instructed that you may
do so by the Invigilator**

(TURN OVER

1 A dislocation with Burgers vector b is pinned at the two endpoints, leaving a segment of length l that is free to glide under a shear stress τ as shown in Fig. 1. As the shear stress τ is gradually increased, the dislocation bows out with a displacement profile $u(x)$.

(a) Describe what is the line tension T of a dislocation and hence explain why the dislocation tends to be straight in the absence of stresses. [15%]

(b) Show that the energy of the bowed segment is given by:

$$E[u(x)] = \int_{-l/2}^{l/2} T\sqrt{1+u'^2} dx - \int_{-l/2}^{l/2} \tau b u(x) dx$$

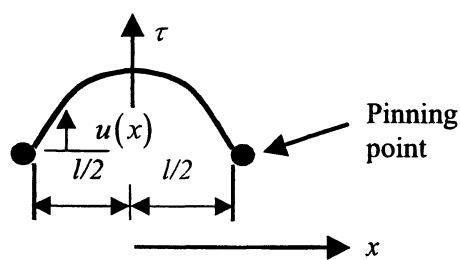
where $u' \equiv du/dx$. [25%]

(c) By solving the differential equation

$$\frac{d}{dx} \left[\frac{Tu'}{\sqrt{1+u'^2}} \right] = -\tau b$$

with the appropriate boundary conditions, show that the displacement profile of the dislocation is given by $u(x) = \sqrt{(T/\tau b)^2 - x^2} - \sqrt{(T/\tau b)^2 - l^2/4}$. [45%]

(d) Show that when the bow-out is small (that is $|u| \ll 1$), the bow-out profile can be approximated by $u(x) = (2\tau b/T)(l^2 - 4x^2)$. [15%]



2 Creep tests are carried out on thin walled cylindrical tubes of radius R and wall thickness t at a temperature of 210°C . The tubes are subjected to an internal pressure p and an axial tensile force F . The tubes are either made of copper or an aluminium alloy, and the magnitude of p and F can be independently varied. For both materials, the creep strain rates $\dot{\epsilon}$ are related to the applied stress σ in a uniaxial test by $\dot{\epsilon} = \dot{\epsilon}_0(\sigma/\sigma_0)^6$ where $\dot{\epsilon}_0$ and σ_0 are reference values. The corresponding times to failure t_f are:

$$t_f = A\sigma^{-6}$$

where A is a constant having the same value for both materials. Under multiaxial stress states and at 210°C , copper fails according to the maximum principal stress criterion whilst the aluminium alloy fails according to the von Mises criterion.

(a) For tubes made of the aluminium alloy, derive the individual components of creep strain rate as functions of the loading parameters p and F . [40%]

(b) Plot for both tubes, the isochronous failure surface in principal stress space. [25%]

(c) Let $\alpha = \sigma_L/\sigma_H$ denote the ratio of the longitudinal stress to the hoop stress and let $I = t_f^{Cu}/t_f^{Al}$ denote the ratio of time to failure for copper and aluminium alloy tubes. Assuming $\alpha \leq 1$, plot I as a function of α , and identify the value of α at which $t_f^{Cu} = t_f^{Al}$ and the values of α at which I reaches its minimum. [35%]

(TURN OVER)

3 (a) Briefly explain the concepts of small scale yielding and large scale yielding in fracture mechanics and discuss the applicability of K_{IC} as a fracture criterion. [20%]

(b) Explain the physical basis of R-curves in metals and the effect of adding fibre reinforcement on the R-curves. [15%]

(c) A semi-infinite crack in a thin sheet is subjected to a concentrated load P at a distance L from the crack tip as shown in Fig. 2. The material tensile yield strength is given as σ_Y . For this situation determine:

(i) The plastic zone size using the Irwin model. [15%]

(ii) The plastic zone size according to the Dugdale model. [25%]

(d) Estimate the plane stress plastic zone size for a semi-infinite mode II crack in an elastic-plastic solid of shear yield strength τ_Y by using the Dugdale model. Give your answer in terms of the mode II stress intensity factor K_{II} . [25%]

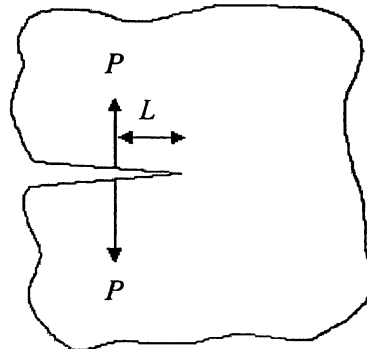


Fig. 2

4 (a) Show that the energy release rate G in a cracked body of compliance C and subjected to a load P is given by

$$G = \frac{1}{2} \frac{P^2}{B} \frac{\partial C}{\partial a}$$

where a is the crack length and B is the thickness of the body. [30%]

(b) A long strip of height $2h$ and thickness B (into the page) with a crack of length $2a$ is subjected to a uniform stress σ along its upper and lower faces as shown in Fig. 3a. For $a \gg h$:

(i) Calculate the strain energy release rate G . [40%]

(ii) Calculate the stress intensity factor K_I assuming plane stress conditions. [10%]

(c) A long, thin-walled circular tube of radius R and wall thickness t contains a crack of length $2a$ through the wall of the tube as sketched in Fig 3b. The crack is inclined at an angle θ to the axis of the tube. Calculate the stress intensity factors K_I and K_{II} for loading comprising an axial force F . [20%]

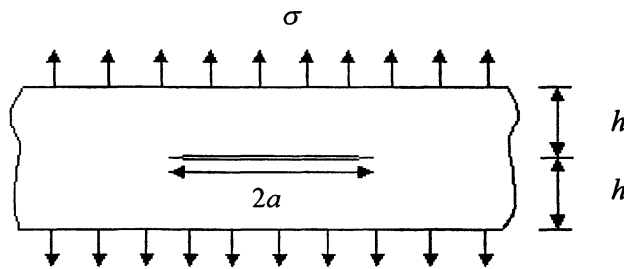


Fig. 3a

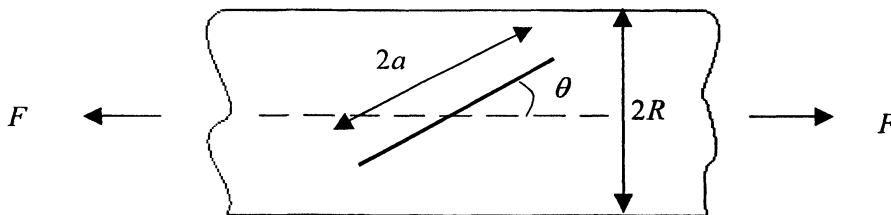


Fig. 3b

END OF PAPER

ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

B1 DEFORMATION AND FRACTURE

FRACTURE MECHANICS DATASHEET

Crack tip plastic zone sizes

$$\text{diameter, } d_p = \begin{cases} \frac{1}{\pi} \left(\frac{K_I}{\sigma_y} \right)^2 & \text{Plane stress} \\ \frac{1}{3\pi} \left(\frac{K_I}{\sigma_y} \right)^2 & \text{Plane strain} \end{cases}$$

Crack opening displacement

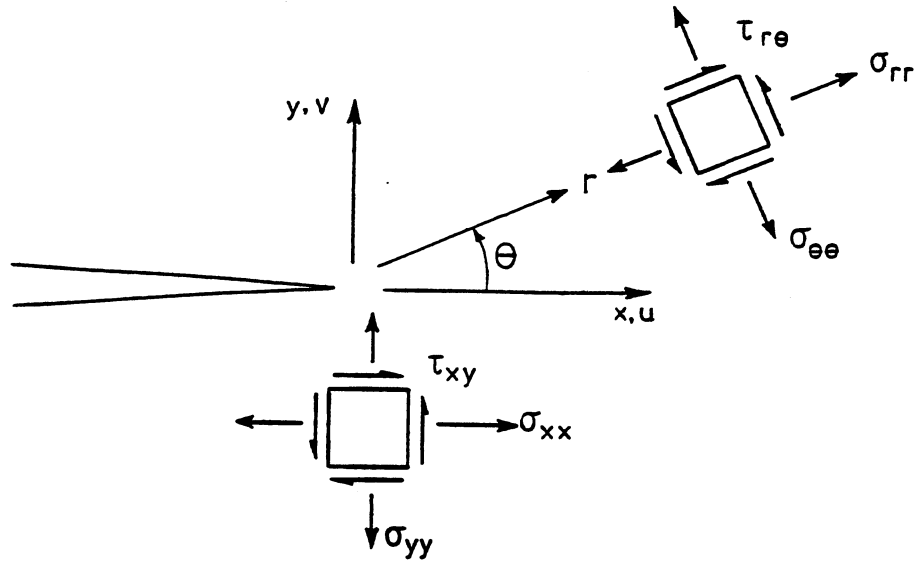
$$\delta = \begin{cases} \frac{K_I^2}{\sigma_y E} & \text{Plane stress} \\ \frac{1}{2} \frac{K_I^2}{\sigma_y E} & \text{Plane strain} \end{cases}$$

Energy release rate

$$G = \begin{cases} \frac{1}{E} K_I^2 & \text{Plane stress} \\ \frac{1 - \nu^2}{E} K_I^2 & \text{Plane strain} \end{cases}$$

Related to compliance C : $G = \frac{1}{2} \frac{P^2}{B} \frac{dC}{da}$

Asymptotic crack tip fields in a linear elastic solid



Mode I

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right)$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right)$$

$$u = \begin{cases} \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \left(\frac{1-\nu}{1+\nu} + \sin^2 \frac{\theta}{2} \right) \cos \frac{\theta}{2} & \text{Plane stress} \\ \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \left(1 - 2\nu + \sin^2 \frac{\theta}{2} \right) \cos \frac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$v = \begin{cases} \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \left(\frac{2}{1+\nu} - \cos^2 \frac{\theta}{2} \right) \sin \frac{\theta}{2} & \text{Plane stress} \\ \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right) \sin \frac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$w = 0$$

Crack tip stress fields (cont'd)

Mode II

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}$$

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{\theta\theta} = -\frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{3}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right)$$

$$u = \begin{cases} \frac{K_{II}}{G} \sqrt{\frac{r}{2\pi}} \left(\frac{2}{1+\nu} + \cos^2 \frac{\theta}{2} \right) \sin \frac{\theta}{2} & \text{Plane stress} \\ \frac{K_{II}}{G} \sqrt{\frac{r}{2\pi}} \left(2 - 2\nu + \cos^2 \frac{\theta}{2} \right) \sin \frac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$v = \begin{cases} \frac{K_{II}}{G} \sqrt{\frac{r}{2\pi}} \left(\frac{\nu-1}{1+\nu} + \sin^2 \frac{\theta}{2} \right) \cos \frac{\theta}{2} & \text{Plane stress} \\ \frac{K_{II}}{G} \sqrt{\frac{r}{2\pi}} \left(-1 + 2\nu + \sin^2 \frac{\theta}{2} \right) \cos \frac{\theta}{2} & \text{Plane strain} \end{cases}$$

$$w = 0$$

Mode III

$$\tau_{zx} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2}$$

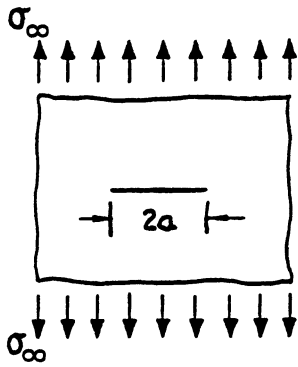
$$\tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos \frac{\theta}{2}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

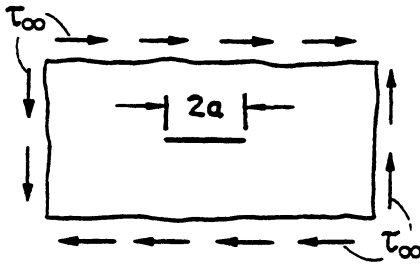
$$w = \frac{K_{III}}{G} \sqrt{\frac{2r}{\pi}} \sin \frac{\theta}{2}$$

$$u = v = 0$$

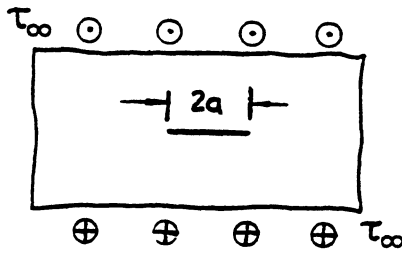
Tables of stress intensity factors



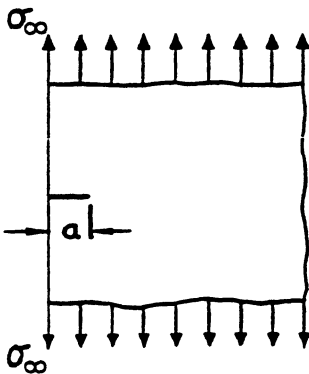
$$K_I = \sigma_{\infty} \sqrt{\pi a}$$



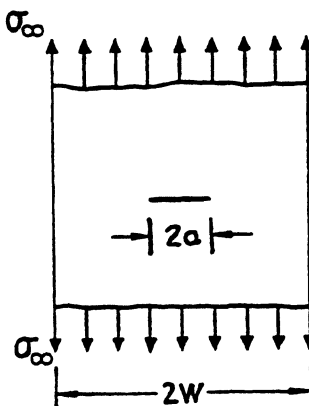
$$K_{II} = \tau_{\infty} \sqrt{\pi a}$$



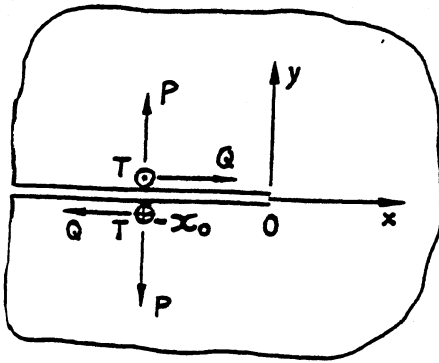
$$K_{III} = \tau_{\infty} \sqrt{\pi a}$$



$$K_I = 1.12 \sigma_{\infty} \sqrt{\pi a}$$



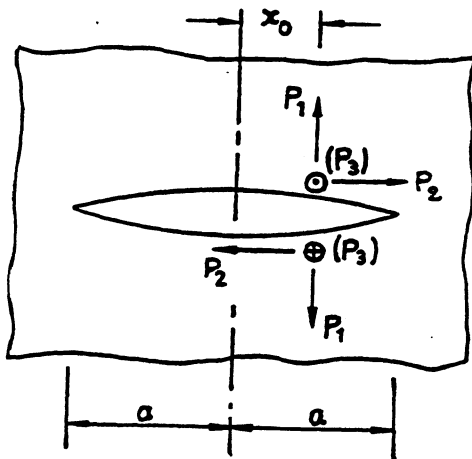
$$K_I = \sigma_{\infty} \sqrt{\pi a} \left(\frac{1 - a/2W + 0.326a^2/W^2}{\sqrt{1 - a/W}} \right)$$



$$K_I = \frac{2P}{\sqrt{2\pi x_0}}$$

$$K_{II} = \frac{2Q}{\sqrt{2\pi x_0}}$$

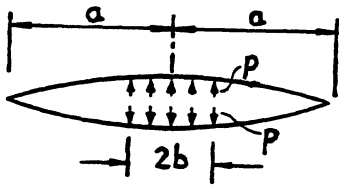
$$K_{III} = \frac{2T}{\sqrt{2\pi x_0}}$$



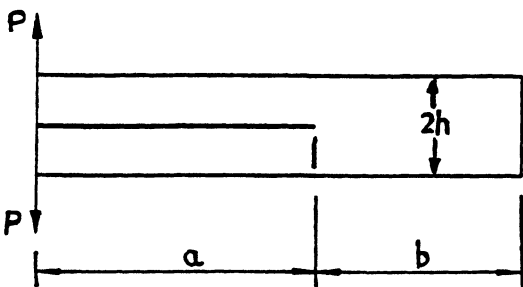
$$K_I = \frac{P_1}{\sqrt{\pi a}} \sqrt{\frac{a+x_0}{a-x_0}}$$

$$K_{II} = \frac{P_2}{\sqrt{\pi a}} \sqrt{\frac{a+x_0}{a-x_0}}$$

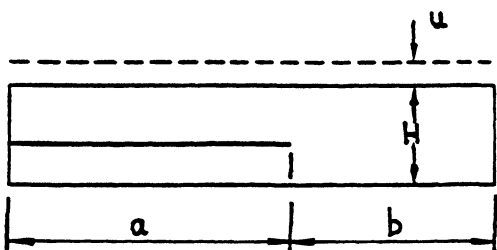
$$K_{III} = \frac{P_3}{\sqrt{\pi a}} \sqrt{\frac{a+x_0}{a-x_0}}$$



$$K_I = \frac{2pb}{\sqrt{\pi a}} \frac{a}{b} \arcsin \frac{b}{a}$$

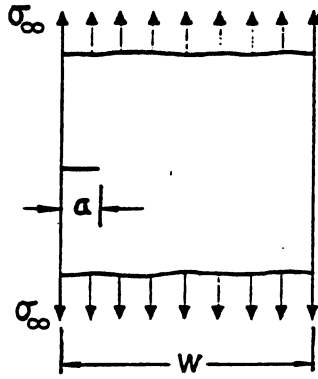


$$K_I = \frac{2\sqrt{3}}{h\sqrt{h}} \frac{Pa}{B} \quad h \ll a \text{ and } h \ll b$$



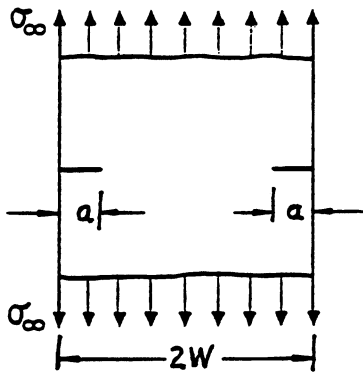
$$K_I = \sqrt{\frac{1}{2\alpha H}} Eu \quad H \ll a \text{ and } H \ll b$$

$$\alpha = \begin{cases} 1 - \nu^2 & \text{Plane stress} \\ 1 - 3\nu^2 - 2\nu^3 & \text{Plane strain} \end{cases}$$

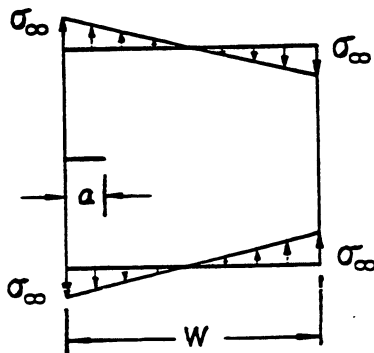


$$a/W < 0.7$$

$$K_I = \sigma_\infty \sqrt{\pi a} \left(1.12 - 0.23 \frac{a}{W} + 10.6 \frac{a^2}{W^2} - 21.7 \frac{a^3}{W^3} + 30.4 \frac{a^4}{W^4} \right)$$

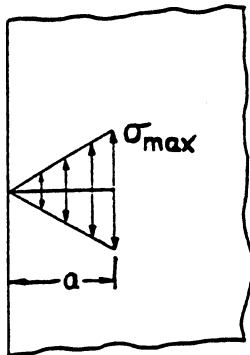


$$K_I = \sigma_\infty \sqrt{\pi a} \left(\frac{1.12 - 0.61a/W + 0.13a^3/W^3}{\sqrt{1 - a/W}} \right)$$

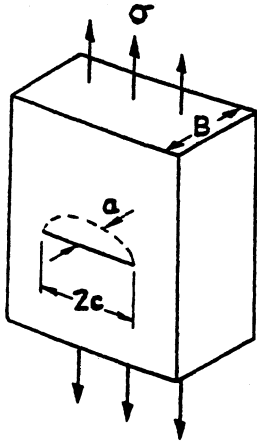


$$a/W < 0.7$$

$$K_I = \sigma_\infty \sqrt{\pi a} \left(1.12 - 1.39 \frac{a}{W} + 7.3 \frac{a^2}{W^2} - 13 \frac{a^3}{W^3} + 14 \frac{a^4}{W^4} \right)$$

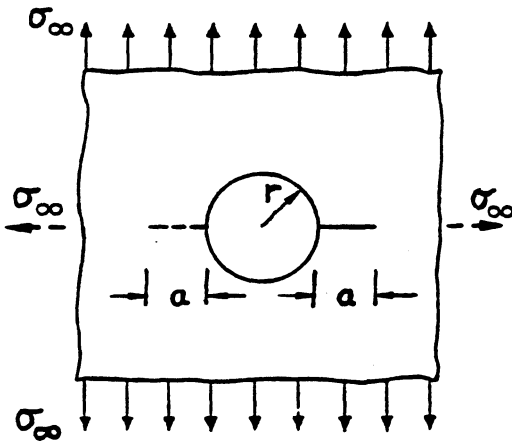
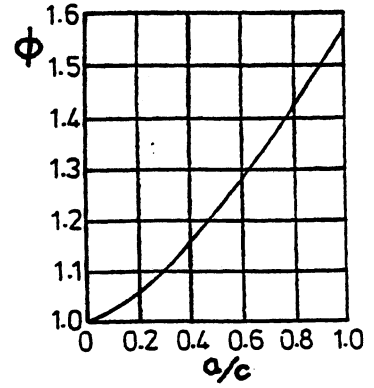


$$K_I = 0.683 \sigma_{\max} \sqrt{\pi a}$$



$$K_I = \frac{1.12}{\Phi} \sigma \sqrt{\pi a}$$

$$\Phi = \int_0^{\pi/2} \left(1 - \frac{c^2 - a^2}{c^2} \sin^2 \theta \right)^{\frac{1}{2}} d\theta$$

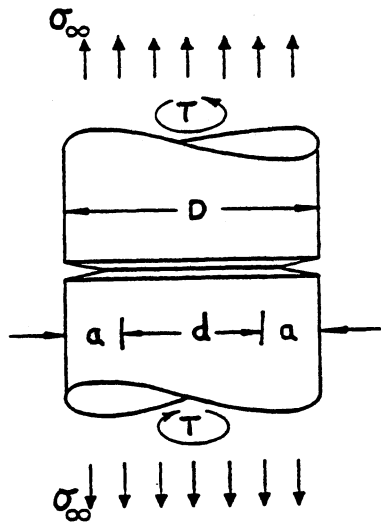


$$K_I = \sigma_{\infty} \sqrt{\pi a} F\left(\frac{a}{r}\right)$$

value of $F(a/r)^\dagger$

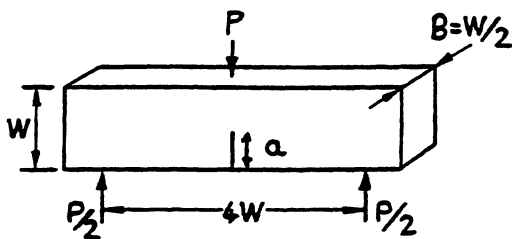
$\frac{a}{r}$	One crack		Two cracks	
	U	B	U	B
0.00	3.36	2.24	3.36	2.24
0.10	2.73	1.98	2.73	1.98
0.20	2.30	1.82	2.41	1.83
0.30	2.04	1.67	2.15	1.70
0.40	1.86	1.58	1.96	1.61
0.50	1.73	1.49	1.83	1.57
0.60	1.64	1.42	1.71	1.52
0.80	1.47	1.32	1.58	1.43
1.0	1.37	1.22	1.45	1.38
1.5	1.18	1.06	1.29	1.26
2.0	1.06	1.01	1.21	1.20
3.0	0.94	0.93	1.14	1.13
5.0	0.81	0.81	1.07	1.06
10.0	0.75	0.75	1.03	1.03
∞	0.707	0.707	1.00	1.00

$^\dagger U = \text{uniaxial } \sigma_{\infty} \quad B = \text{biaxial } \sigma_{\infty}.$

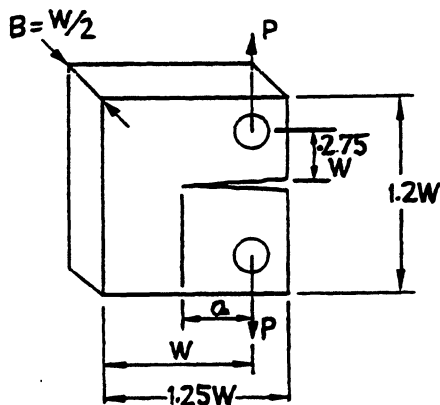


$$K_I = \sigma_{\infty} \sqrt{\pi a} \left(\frac{D}{d} + \frac{1}{2} + \frac{3d}{8D} - 0.36 \frac{d^2}{D^2} + 0.73 \frac{d^3}{D^3} \right) \frac{1}{2} \sqrt{\frac{D}{d}}$$

$$K_{III} = \frac{16T}{\pi D^3} \sqrt{\pi a} \left(\frac{D^2}{d^2} + \frac{1}{2} \frac{D}{d} + \frac{3}{8} + \frac{5d}{16D} + \frac{35d^2}{128D^2} + 0.21 \frac{d^3}{D^3} \right) \frac{3}{8} \sqrt{\frac{D}{d}}$$



$$K_I = \frac{4P}{B} \sqrt{\frac{\pi}{W}} \left\{ 1.6 \left(\frac{a}{W} \right)^{1/2} - 2.6 \left(\frac{a}{W} \right)^{3/2} + 12.3 \left(\frac{a}{W} \right)^{5/2} - 21.2 \left(\frac{a}{W} \right)^{7/2} + 21.8 \left(\frac{a}{W} \right)^{9/2} \right\}$$



$$K_I = \frac{P}{B} \sqrt{\frac{\pi}{W}} \left\{ 16.7 \left(\frac{a}{W} \right)^{1/2} - 104.7 \left(\frac{a}{W} \right)^{3/2} + 369.9 \left(\frac{a}{W} \right)^{5/2} - 573.8 \left(\frac{a}{W} \right)^{7/2} + 360.5 \left(\frac{a}{W} \right)^{9/2} \right\}$$

Paper G4: Mechanics of Solids
ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	<u>Discs and tubes</u>	<u>Spheres</u>
Equilibrium	$\sigma_{\theta\theta} = \frac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \frac{1}{2r} \frac{d(r^2\sigma_{rr})}{dr}$
Lam's equations (in elasticity)	$\sigma_{rr} = A - \frac{B}{r^2} - \frac{3+\nu}{8} \rho\omega^2 r^2 - \frac{E\alpha}{r^2} \int_c^r rTdr$	$\sigma_{rr} = A - \frac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \frac{B}{r^2} - \frac{1+3\nu}{8} \rho\omega^2 r^2 + \frac{E\alpha}{r^2} \int_c^r rTdr - E\alpha T$	$\sigma_{\theta\theta} = A + \frac{B}{2r^3}$

2. Plane stress and plane strain

	<u>Cartesian coordinates</u>	<u>Polar coordinates</u>
Strains	$\epsilon_{xx} = \frac{\partial u}{\partial x}$ $\epsilon_{yy} = \frac{\partial v}{\partial y}$ $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$	$\epsilon_{rr} = \frac{\partial u}{\partial r}$ $\epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}$ $\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}$
Compatibility	$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}$	$\frac{\partial}{\partial r} \left\{ r \frac{\partial \gamma_{r\theta}}{\partial \theta} \right\} = \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial \epsilon_{\theta\theta}}{\partial r} \right\} - r \frac{\partial \epsilon_{rr}}{\partial r} + \frac{\partial^2 \epsilon_{rr}}{\partial \theta^2}$
or (in elasticity)	$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} (\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right\} (\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$ $\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$	$\frac{\partial}{\partial r} (r\sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$ $\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r} (r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4 \phi = 0$ (in elasticity)	$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} \left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right\} = 0$	$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right\} \times \left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right\} = 0$
Airy Stress Function	$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$ $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$ $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$ $\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$ $\sigma_{r\theta} = -\frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx} (= \tau_x) = \frac{dF}{dy}$, $\sigma_{zy} (= \tau_y) = -\frac{dF}{dx}$

Equilibrium: $T = 2 \int_A F dA$

Governing equation for elastic torsion: $\nabla^2 F = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2} \int_V \underline{\underline{\varepsilon}}^T [D] \underline{\underline{\varepsilon}} dV$, $W = \underline{\underline{p}}^T \underline{\underline{u}}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_p , can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_p & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_p \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_p .

Expanding: $\sigma_p^3 - I_1 \sigma_p^2 + I_2 \sigma_p - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2} \{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \}}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3} \{ d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2 \}}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma_1} = \frac{d\varepsilon_2}{\sigma_2} = \frac{d\varepsilon_3}{\sigma_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2} \frac{d\bar{\varepsilon}}{\bar{\sigma}}$$