UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Tuesday 20 April 2004 2.30 to 4

Module 4C2

DESIGNING WITH COMPOSITES

Answer not more than **three** *questions.*

All questions carry the same number of marks.

The **approximate** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Module 4C2 datasheet (6 pages).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 A $[\pm 45]_S$ laminate plate is autoclaved from 4 plies, each of thickness t. Each ply has the following elasticity constants: $E_2 = 0.1\,E_1$, $G_{12} = E_1/9.99$ and $\nu_{12} = 0.1$.

(a) Determine the components of the laminate extensional matrix $[A]$, the laminate coupling stiffness matrix $[B]$ and the laminate bending stiffness matrix $[D]$ for the plate. [60%]

(b) The plate is cut into a beam of width w and length ℓ, with the plies still at ±45° orientations relative to the edges of the beam. The beam is loaded along its length as a pin-jointed strut in compression. Obtain an expression for the Euler buckling load in terms of the above material and geometric properties. Use may be made of the Structures Data Book as appropriate. [10%]

(c) Discuss the practical problems of incorporating such a strut into the design of a lightweight structure, anticipate the potential failure modes, and suggest how the column design might be improved. [30%]

2 (a) Explain why strain allowables are a useful way to model the strength of composite laminates and how they can be used to facilitate preliminary design of composite structures. [20%]

(b) Consider the preliminary design of a square laminate of side 2 metres made from 0°, ±45° and 90° plies of carbon fibre epoxy material, each of thickness 0.125 mm. The plate is subjected to an axial line load N_x = 0.3 MNm^{-1} and a shear line load N_{xy} = 0.2 MNm^{-1}. Figure 1 gives suitable carpet plots for the material. **A copy of this figure has been supplied and should be handed in with the answer.** A performance index f for the component is defined as

$$f = 2m + \delta$$

where m is the mass of the component in kilograms and δ is the axial extension of the component, in millimetres. The costs of material and manufacture may be neglected. Estimate a suitable laminate lay-up which minimises the performance index f while still supporting the loads. [80%]

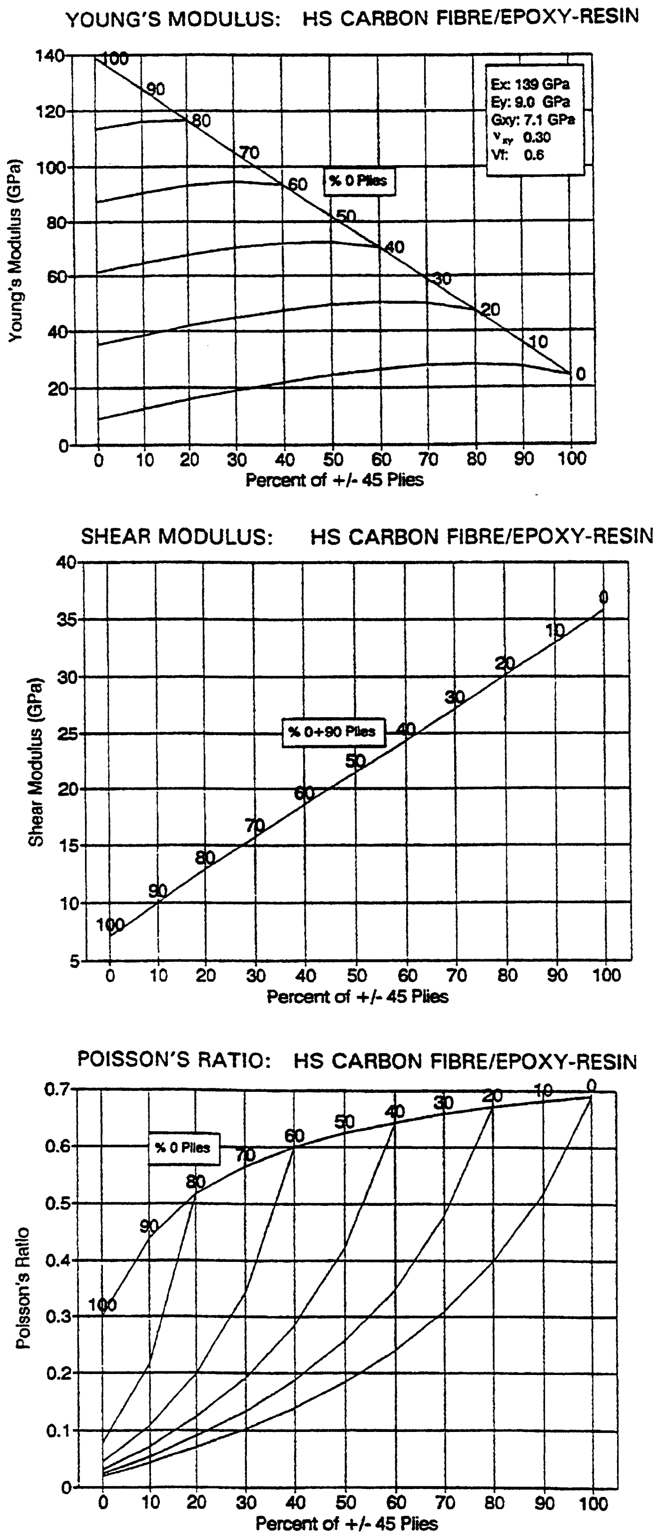


Fig. 1

(TURN OVER

3 A $[0/\pm\theta]_S$ laminate made from six plies of AS/3501 carbon fibre epoxy material (material data on the data sheet) is subjected to uniform tension along the 0° direction. The $[Q]$ matrix for a 0° lamina and the $[\bar{Q}]$ matrix for a 30° lamina are as follows:

$$[Q] = \begin{bmatrix} 139 & 2.7 & 0 \\ 2.7 & 9.0 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{GPa}, \quad [\bar{Q}_{30}] = \begin{bmatrix} 85 & 24 & 41 \\ 24 & 20 & 16 \\ 41 & 16 & 28 \end{bmatrix} \text{GPa}$$

(a) Explain in detail how failure of the laminate can be predicted using laminate plate theory and the Tsai-Hill failure criterion. Do not undertake detailed calculations at this stage. [30%]

(b) Find the first ply failure stress of the laminate predicted by the Tsai-Hill failure criterion, for $\theta = 30°$. [45%]

(c) For the above $[0/\pm\theta]_S$ laminate loaded in tension along the 0° direction, sketch the theoretical variation with ply angle θ of the first ply failure stress, according to the Tsai-Hill criterion, estimating the values of any salient points. How would you expect experimental measurements of ultimate failure stress to compare with these predictions? [25%]

4 Explain in detail and comment on the following observations.

(a) Load bearing lightweight structures make increasing use of carbon fibre laminates with a tough epoxy matrix, despite the fact that these laminates have a reduced compressive strength. [25%]

(b) Torque shafts are made from composite tubes with a combination of off-axis and axial fibres. [25%]

(c) Yacht hulls are made from autoclaved carbon fibre laminate parts rather than chopped strand mat of short glass fibres in a polyester matrix. In contrast, leisure boats are often made from chopped strand mat. [25%]

(d) Sandwich beams in 3 point bending commonly fail at lower loads than that given by the bending strength of the beam. [25%]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix} = [S]\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} \quad \text{where } [S] = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12}\end{bmatrix}$$

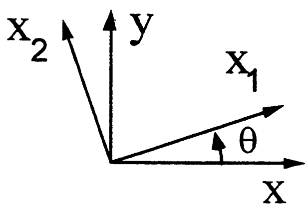
[S] is symmetric, giving $\nu_{12}/E_1 = \nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix}$$

where
$$\begin{aligned} Q_{11} &= E_1/(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ, with respect to the (x, y) axes.



Then,
$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} = [T]\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix} = [T]^{-T}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{pmatrix}$$

where
$$[\mathrm{T}] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

and
$$[\mathrm{T}]^{-T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\overline{Q}]$ when the axes are rotated from (x_1, x_2) to (x, y)

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \overline{Q}_{11} &= Q_{11}C^4 + Q_{22}S^4 + 2(Q_{12} + 2Q_{66})S^2C^2 \\ \overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})S^2C^2 + Q_{12}(C^4 + S^4) \\ \overline{Q}_{22} &= Q_{11}S^4 + Q_{22}C^4 + 2(Q_{12} + 2Q_{66})S^2C^2 \\ \overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})SC^3 - (Q_{22} - Q_{12} - 2Q_{66})S^3C \\ \overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})S^3C - (Q_{22} - Q_{12} - 2Q_{66})SC^3 \\ \overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})S^2C^2 + Q_{66}(S^4 + C^4) \end{aligned}$$

with $C = \cos\theta$ and $S = \sin\theta$.

The compliance matrix $[S] \equiv [Q]^{-1}$ transforms to $[\overline{S}] \equiv [\overline{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1, x_2) to (x, y), as

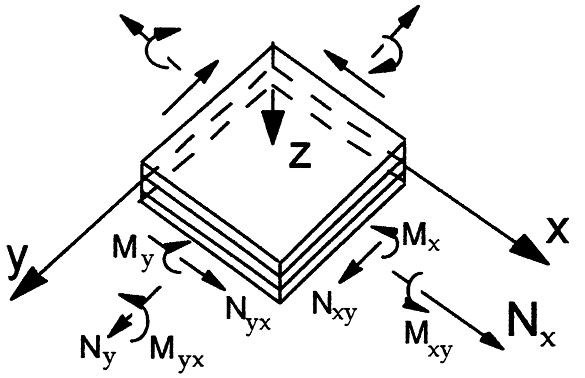
$$[\overline{S}] = [T]^T [S][T]$$

and in component form,

$$\begin{aligned} \overline{S}_{11} &= S_{11}C^4 + S_{22}S^4 + (2S_{12} + S_{66})S^2C^2 \\ \overline{S}_{12} &= S_{12}(C^4 + S^4) + (S_{11} + S_{22} - S_{66})S^2C^2 \\ \overline{S}_{22} &= S_{11}S^4 + S_{22}C^4 + (2S_{12} + S_{66})S^2C^2 \\ \overline{S}_{16} &= (2S_{11} - 2S_{12} - S_{66})SC^3 - (2S_{22} - 2S_{12} - S_{66})S^3C \\ \overline{S}_{26} &= (2S_{11} - 2S_{12} - S_{66})S^3C - (2S_{22} - 2S_{12} - S_{66})SC^3 \\ \overline{S}_{66} &= (4S_{11} + 4S_{22} - 8S_{12} - 2S_{66})S^2C^2 + S_{66}(C^4 + S^4) \end{aligned}$$

with $C = \cos\theta$, $S = \sin\theta$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $\left(\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0\right)^T$ and to a curvature $\left(\kappa_x, \kappa_y, \kappa_{xy}\right)^T$. The stress resultants $\left(N_x, N_y, N_{xy}\right)^T$ and bending moment per unit length $\left(M_x, M_y, M_{xy}\right)^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & \cdot & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^0 \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\overline{Q}_{ij}\right)_k dz = \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\overline{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\overline{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts $i, j = 1, 2$ or 6.

Here,

t = laminate thickness

z_{k-1} = distance from middle surface to the inner surface of the k - th lamina

z_k = distance from middle surface to the outer surface of the k - th lamina

Quadratic failure criteria.

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\dfrac{\sigma_1^2}{s_L^2} - \dfrac{\sigma_1\sigma_2}{s_L^2} + \dfrac{\sigma_2^2}{s_T^2} + \dfrac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \dfrac{1}{s_L^+ s_L^-}$, $F_{22} = \dfrac{1}{s_T^+ s_T^-}$, $F_1 = \dfrac{1}{s_L^+} - \dfrac{1}{s_L^-}$, $F_2 = \dfrac{1}{s_T^+} - \dfrac{1}{s_T^-}$, $F_{66} = \dfrac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{(F_{11}F_{22})^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

M. P. F. Sutcliffe
N. A. Fleck
October 2002