UNIVERSITY OF
CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Wednesday 21 April 2004 2:30 to 4

Module 4C9

CONTINUUM MECHANICS

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

Attachments:
i) Special datasheet (2 pages).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

(TURN OVER

1 (a) A sinusoidal pressure distribution $p(x) = p_0 \cos(\pi x / L)$ acts on one edge of an infinitely large plate. On this edge, the shear stress is zero.

(i) For an Airy's stress function of the form $\Phi = f(y)\cos(\pi x / L)$, obtain an expression for $f(y)$ that satisfies the equations of equilibrium and compatibility. [25%]

(ii) Obtain expressions for stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ near the edge of the plate. [25%]

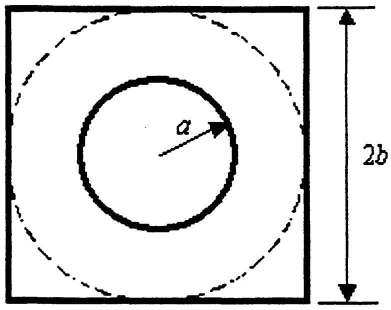

Fig. 1

(b) Figure 2 shows a square plate containing a central hole of radius a; the side length and thickness of the plate are $2b$ ($b > a$) and t, respectively. A uniform pressure p applied to the surface of the hole is gradually increased until the collapse pressure p^L is reached. The plate is composed of an incompressible and rigid, perfectly-plastic material that has shear yield stress τ_y and yields according to Tresca's criterion.

(i) Consider a cylinder bounded by the inner surface of radius a and the external surface of radius b (represented by the dashed circle in Fig. 1). Assuming plane strain and stress-free boundary condition at the external surface, show that, after yielding has occurred, a statically admissible field in the cylinder is given by:

$$\sigma_{rr} = -p + \sigma_0 \ln(r/a), \quad \sigma_{\theta\theta} = -p + \sigma_0\left[1 + \ln(r/a)\right], \quad \sigma_{r\theta} = 0, \quad \text{for } a \le r \le d$$

where $a \le r \le d$ is the yielding zone and $d \le b$. Hence obtain a lower bound solution of the collapse pressure p^L for the square plate. [30%]

(ii) How would your solution to p^L change if the plate satisfies plane stress conditions? [20%]

2. Consider a thin-walled cylinder subjected to a combined loading of torsion and longitudinal tension. A infinitely small material element taken from the wall of the cylinder has longitudinal stress σ_{zz}, shear stress $\sigma_{\theta z}$, longitudinal strain ε_{zz} and engineering shear strain $\gamma_{\theta z}(\equiv 2\varepsilon_{\theta z})$ in the cylindrical coordinate system. The material of the cylinder is incompressible and elastic-perfectly plastic, with Young's modulus E, shear modulus G and tensile yield stress σ_y, and yields according to the Mises criterion.

(a) By introducing the following non-dimensional variables:

$$\sigma = \frac{\sigma_{zz}}{\sigma_y}, \quad \tau = \frac{\sigma_{\theta z}}{\tau_y}, \quad \varepsilon = \frac{\varepsilon_{zz}}{\varepsilon_y}, \quad \gamma = \frac{\gamma_{\theta z}}{\gamma_y}$$

where $\tau_y = \sigma_y / \sqrt{3}$ is the yield stress in shear, $\varepsilon_y = \sigma_y / E$ and $\gamma_y = \tau_y / G$, show that the Mises yield criterion for the cylinder can be written as

$$\sigma^2 + \tau^2 = \varepsilon^2 + \gamma^2 = 1$$ [25%]

(b) Using the Levy-Mises constitutive relations $d\varepsilon_{ij}^p = s_{ij} d\lambda$ and the yield criterion derived in (a), show that:

$$\frac{d\varepsilon - d\sigma}{d\gamma - d\tau} = \frac{\sigma}{\tau} \quad \text{and} \quad \frac{d\sigma}{d\varepsilon} = \sqrt{1-\sigma^2}\left(\sqrt{1-\sigma^2} - \sigma\frac{d\gamma}{d\varepsilon}\right)$$ [50%]

(cont.

(c) The loading of the cylinder is displacement-controlled. Consider the deformation history from a to b as shown below in Fig. 2:

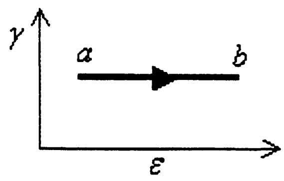

Fig. 2

Determine the relationship at b between axial strain ε and stress σ if at deformation state a, the stresses and strains are as follows: $\sigma = \sigma_0$, $\tau = \tau_0$, $\varepsilon = \varepsilon_0$ and $\gamma = \gamma_0$. [25%]

(Hint: $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1$)

3. Figure 3 shows a thin square plate with uniform thickness, density ρ and sides of length 2a. The plate rotates at angular speed Ω about the horizontal centreline $y = 0$. All edges and surfaces are traction free.

(a) Write an expression for the complete stress field in the rotating plate. [20%]

(b) Transform your solution to polar coordinates with origin at the centre and give expressions for the state of stress at an arbitrary point r, θ. For a centred circular disk of radius βa, $0 < \beta < 1$, inscribed in the square, find the traction $\mathbf{t}(\beta a, \theta)$ given by the stress field from part (a) evaluated at the edge of the disk. Show that the components of the traction can be expressed as

$$\sigma_{rr}(\beta a, \theta) = \alpha\left[\left(4 - 3\beta^2\right) - 4\left(1 - \beta^2\right)\cos 2\theta - \beta^2 \cos 4\theta\right],$$

$$\sigma_{r\theta}(\beta a, \theta) = \alpha\left[\left(4 - 2\beta^2\right)\sin 2\theta + \beta^2 \sin 4\theta\right]$$

where you are to obtain an expression for the constant α. [35%]

(c) For a thin square plate with side length $2a$, a central hole with radius βa and no body force, find a stress field that satisfies the traction boundary condition in part (b), without regard for the external boundary. [30%]

(d) Explain whether superposition of the solutions to parts (a) and (c) can be used to obtain the stress field in a square plate with a central circular hole when the plate is rotating about a diametral centreline $\theta = 0$ at angular speed Ω. If this is not a complete solution, what else is required? [15%]

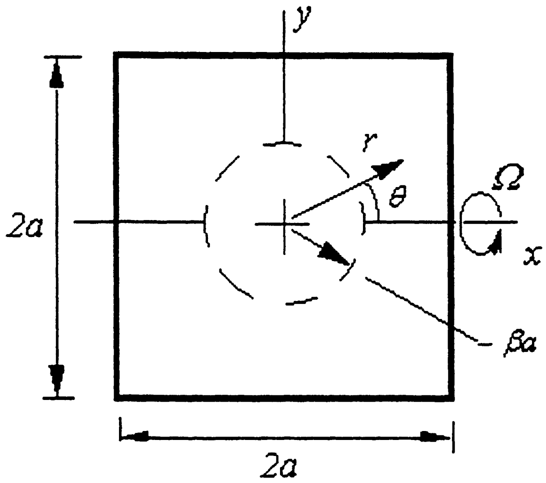

Fig. 3

END OF PAPER

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POLAR (CYLINDRICAL) COORDINATES

TRANSFORMATION: Cartesian — Cylindrical

$$x = r\cos\theta, \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2, \qquad \theta = \arctan(y/x)$$

notice that

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

hence equil. and compatibility satisfied if $\nabla^4\phi = 0$ where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta - 2\sigma_{xy}\sin\theta\cos\theta$$
$$= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$
$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

Displacement components and strain-disp. relations

$$u_x = u_r\cos\theta - u_\theta\sin\theta$$
$$u_y = u_r\sin\theta + u_\theta\cos\theta$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad \varepsilon_{r\theta} = \left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

HARMONIC FUNCTIONS:

$$\phi = \sum_{n=0}^{\infty} f_n(r)\cos n\theta + \sum_{n=1}^{\infty} g_n(r)\sin n\theta$$
$$0 = \nabla^4\phi \quad \Rightarrow \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)^2 f_n = 0$$

has solution

$$f_0 = A_{01}r^2 + A_{02}r^2\ln r + A_{03}\ln r + A_{04}$$
$$f_1 = A_{11}r^3 + A_{12}r\ln r + A_{13}r + A_{14}$$
$$f_n = A_{n1}r^{n+2} + A_{n2}r^{-n+2} + A_{n3}r^n + A_{n4}r^{-n}, \qquad n = 2,3,\ldots$$

Table 8.1: The Michell solution — stress components

coeff.	ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
a_0	r^2	2	0	2
b_0	$r^2 \log r$	$2\log r + 1$	0	$2\log r + 3$
c_0	$\log r$	$1/r^2$	0	$-1/r^2$
d_0	θ	0	$1/r^2$	0
a_1	$r^3 \cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
b_1	$r\theta\sin\theta$	$2\cos\theta/r$	0	0
c_1	$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
d_1	$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
a_1'	$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
b_1'	$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
c_1'	$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
d_1'	$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
a_2	$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$	$(n+1)(n+2)r^n\cos n\theta$
b_2	$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n}\cos n\theta$	$-n(n-1)r^{-n}\sin n\theta$	$(n-1)(n-2)r^{-n}\cos n\theta$
	$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
	$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2}\cos n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$
	$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$	$(n+1)(n+2)r^n\sin n\theta$
	$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n}\sin n\theta$	$n(n-1)r^{-n}\cos n\theta$	$(n-1)(n-2)r^{-n}\sin n\theta$
	$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
	$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$	$n(n+1)r^{-n-2}\sin n\theta$

Around hole, single valued disp. requires $u_\theta(r,0) = u_\theta(r,2\pi)$; hence,

plane strain: $b_0 = 0$, $b_1(\kappa - 1) = -c_1(\kappa + 1)$, $b_1'(\kappa - 1) = c_1'(\kappa + 1)$

plane stress: $b_0 = 0$, $b_1 = -2c_1/(1-\nu)$, $b_1' = 2c_1'/(1-\nu)$

Timoshenko & Goodier, 3rd ed. p135

Table 9.1: The Michell solution — displacement components

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ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa - 1)r$	0
$r^2\log r$	$(\kappa - 1)r\log r - r$	$(\kappa + 1)r\theta$
$\log r$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa - 2)r^2\cos\theta$	$(\kappa + 2)r^2\sin\theta$
$r\theta\sin\theta$	$\frac{1}{2}[(\kappa - 1)\theta\sin\theta - \cos\theta + (\kappa + 1)\log r\cos\theta]$	$\frac{1}{2}[(\kappa - 1)\theta\cos\theta - \sin\theta - (\kappa + 1)\log r\sin\theta]$
$r\log r\cos\theta$	$\frac{1}{2}[(\kappa + 1)\theta\sin\theta - \cos\theta + (\kappa - 1)\log r\cos\theta]$	$\frac{1}{2}[(\kappa + 1)\theta\cos\theta - \sin\theta - (\kappa - 1)\log r\sin\theta]$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa - 2)r^2\sin\theta$	$-(\kappa + 2)r^2\cos\theta$
$r\theta\cos\theta$	$\frac{1}{2}[(\kappa - 1)\theta\cos\theta + \sin\theta - (\kappa + 1)\log r\sin\theta]$	$\frac{1}{2}[-(\kappa - 1)\theta\sin\theta - \cos\theta - (\kappa + 1)\log r\cos\theta]$
$r\log r\sin\theta$	$\frac{1}{2}[-(\kappa + 1)\theta\cos\theta - \sin\theta + (\kappa - 1)\log r\sin\theta$	$\frac{1}{2}[(\kappa + 1)\theta\sin\theta + \cos\theta + (\kappa - 1)\log r\cos\theta]$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa - n - 1)r^{n+1}\cos n\theta$	$(\kappa + n + 1)r^{n+1}\sin n\theta$
$r^{-n+2}\cos n\theta$	$(\kappa + n - 1)r^{-n+1}\cos n\theta$	$-(\kappa - n + 1)r^{-n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{-n}\cos n\theta$	$nr^{-n-1}\cos n\theta$	$nr^{-n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa - n - 1)r^{n+1}\sin n\theta$	$-(\kappa + n + 1)r^{n+1}\cos n\theta$
$r^{-n+2}\sin n\theta$	$(\kappa + n - 1)r^{-n+1}\sin n\theta$	$(\kappa - n + 1)r^{-n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{-n}\sin n\theta$	$nr^{-n-1}\sin n\theta$	$-nr^{-n-1}\cos n\theta$

For plane strain

$$\kappa = 3 - 4\nu$$

whilst for plane stress

$$\kappa = \left(\frac{3 - \nu}{1 + \nu}\right)$$