UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Wednesday 21 April 2004 9 to 10.30

Module 4C12

WAVE PROPAGATION

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

(TURN OVER

1 (a) A canal has deep water confined between parallel vertical walls a distance L apart. Cartesian axes are aligned as shown in Fig. 1. Write down the governing differential equation and boundary conditions for surface waves of small amplitude on the water, under the same assumptions made in the lectures. What physical effects are omitted from this approximate model? [20%]

(b) Use the method of separation of variables to find possible travelling-wave solutions in which the dependence on x and time t takes the form $e^{i(wt-kx)}$. Show that the dispersion relation for these waves can be written

$$k^2 = \frac{\omega^4}{g^2} - \left[\frac{n\pi}{L}\right]^2$$

where n can take certain values, to be specified. [40%]

(c) For each possible wave type, calculate the cut-on frequency (if applicable), and the phase and group velocity as a function of frequency ω. [20%]

(d) Sketch the form of the waves for the first three waveguide modes. On a single graph, plot the phase velocity against frequency for the first three waveguide modes. Plot a similar graph showing the group velocities. In both cases, assume that the width of the canal is 5 m, and plot on a frequency scale 0–1 Hz. [20%]

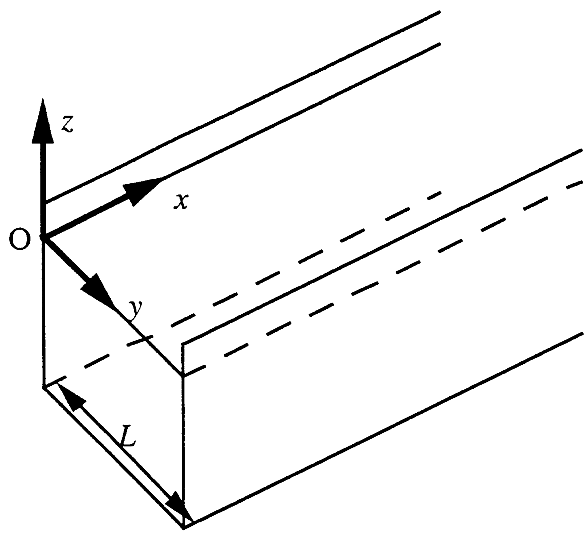

Fig. 1

2 A slender elastic wire with length L, cross-sectional area A, wave speed c_0, and failure stress σ_F hangs vertically from a rigid clamp. Hanging at the bottom of the wire is a perfectly-plastic block of mass M that acts as a stop. A rigid weight of mass M is threaded onto the wire and falls some distance before it collides against the stop with impact speed $\tilde{V}_0$ at time $t = 0$. Let x be distance downward along the wire measured from the clamp.

(a) Write an expression for the initial speed V_0 of the combined falling mass and block at time $t = 0+$ in relation to the impact speed $\tilde{V}_0$. Also write an expression for the static stress in the wire before impact, $\sigma_0 = \sigma(x,0)$. [10%]

(b) Show that for $t > 0$, the combined weight and stop have velocity $V(t)$ that can be expressed as

$$V(t) = \alpha + (\rho c_0 V_0 - \alpha) e^{-\beta t}$$

where you are to write explicit expressions for the constants α, β. Give an expression for the period of time t for which this expression is applicable. [35%]

(c) Write expressions for the stress distribution $\sigma(x,t)$ in the wire $0 < x \leq L$ during $0 < t < L / c_0$. Sketch the distribution of stress at time $t = L / 2c_0$. [30%]

(d) Write an expression for the minimum impact speed $\tilde{V}_{0F}$ required to break the wire during the period $0 < t < 2L / c_0$. Describe the location x_F of the break and the time after impact t_F when it occurs. [25%]

3 Beans are channeled to flow past a "bean counter" located at $x = 0$. The density of beans $\rho(x,t)$ varies with both time and spatial position along the channel. The velocity of beans along the channel $V(x,t)$ is a maximum V_m when the density is negligible but this velocity falls to zero if the density equals ρ_m. In general the velocity of beans varies in accord with $V/V_m = 1 - \sqrt{\rho/\rho_m}$.

(a) Find the density of beans which gives the fastest rate of flow past the counter. [15%]

(b) Suppose that initially a semi-infinite column of beans is constrained to move uniformly with an initial spatial distribution

$$\rho(x,t) = \begin{cases} (25/36)\rho_m, & x < 0 \\ 0, & x > 0 \end{cases}$$

At time $t = 0$ the constraint is removed and thereafter each bean moves at speed $V(\rho)$.

Sketch a characteristic diagram of the region showing the head of the column and label characteristic lines that represent (i) the position of the leading bean as a function of time, and (ii) the spatial extent of that part of the column which is changing velocity in reaction to removal of the constraint. [25%]

(c) Show that in the fan of characteristic lines that emanate from the head of the column when the constraint is removed, the position of any bean $\xi(t)$ satisfies an equation

$$3t\frac{d\xi}{dt} - 2\xi = V_m t$$ [35%]

(d) For a bean that is initially a distance ξ_0 behind the head of the column, find the position $\xi(t)$. Sketch the path of this typical bean as a dashed line on your characteristic diagram drawn for part (b). [25%]

END OF PAPER