ENGINEERING TRIPOS PART IIB

Wednesday 21 April 2004 2.30 to 4

Module 4F1

CONTROL SYSTEM DESIGN

Answer not more than **two** *questions.*

All questions carry the same number of marks.

The **approximate** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Formulae sheet (3 pages).

Supplementary pages:
Two extra copies of Fig. 1 (Question 3).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Let $S(s)$ denote the sensitivity function of a feedback system with return ratio $L(s)$. Suppose $L(s)$ is stable, minimum phase and has at least second order roll-off at high frequencies.

(a) Show that [25%]

$$\int_0^\infty \ln|S(j\omega)|d\omega = 0.$$

(b) Suppose that $|S(j\omega)| \leq M$ for all ω, where $M > 1$. Give a graphical interpretation of this condition in terms of $L(s)$. Show that the condition guarantees a phase margin of at least:

$$2\sin^{-1}\left(\frac{1}{2M}\right).$$

Can a lower bound on the phase margin be used to determine an upper bound on M? [25%]

(c) Suppose the following specifications are required:

A: $|S(j\omega)| \leq 0.1$ for $\omega < 1$;
B: $|S(j\omega)| \leq M$ for all ω;
C: $L(s)$ has gain smaller than that of the system $1/(s+1)^2$ for $\omega \geq 10$.

Show that specification C implies that:

$$\int_{10}^\infty \ln|S(j\omega)|d\omega < 0.1.$$

[Hint: you may assume that [25%]

$$\int \ln\left(1-(1+\omega^2)^{-1}\right)d\omega = \omega\ln\left(1-(1+\omega^2)^{-1}\right) - 2\tan^{-1}\omega.]$$

(d) Hence find an M_0 so that the specifications in Part (c) are infeasible for $M < M_0$. [25%]

2 (a) Let $R(s) = N(s)/D(s)$ be a stable rational transfer function with $R(0) = 1$, $R(1) = 0$ and $\deg(N) < \deg(D)$.

(i) By considering the definition of the Laplace transform, or otherwise, show that the output $y(t)$ of $R(s)$ in response to a step input must satisfy:

$$\int_0^\infty y(t)e^{-t}dt = 0.$$

What conclusion can you draw about the necessity of undershoot for such a transfer function? [20%]

(ii) Find a condition on $m = \deg(D) - \deg(N)$ for the initial slope of the step response of $R(s)$ to be zero. [15%]

(b) Consider a plant with transfer function

$$\frac{s-1}{s^2 - 5s + 6}.$$

(i) Sketch the root-locus for this plant, and hence show that it can be stabilised by constant gain feedback. Find the range of stabilising gains. [30%]

(ii) Design a control scheme for this plant to achieve internal stability, unity d.c. gain from reference input $r(t)$ to plant output $y(t)$ and zero initial slope of this step response. [25%]

(iii) Without making any detailed calculations, sketch the form of $y(t)$ in response to a step input at $r(t)$ for your design. [10%]

3 Figure 1 is the Bode diagram of a system $G(s)$ for which a feedback compensator $K(s)$ is to be designed. It may be assumed that $G(s)$ is a real-rational transfer function.

(a) (i) Sketch on a copy of Fig. 1 the expected phase of $G(j\omega)$ if $G(s)$ had no poles or zeros with $\text{Re}(s) > 0$; [10%]

(ii) Explain why the sketch only allows the *least* number of possible right half plane poles of $G(s)$ to be determined. What is this least number? [20%]

(iii) Comment on any limitations on the achievable crossover frequency that might be faced in a control systems design for this plant. [10%]

(b) Assume that $G(s)$ has the least number of right half plane poles that you determined in Part (a)(ii). A feedback compensator $K(s)$ is required to provide internal stability of the closed-loop system and a phase margin of at least 50°. Show that this can be achieved using a $K(s)$ with one pole and one zero. Use a Nyquist diagram sketch to justify your conclusion. [30%]

(c) The compensator obtained in Part (b) is to be modified to achieve the following specification:

$$\text{A: } |S(j\omega)| < 0.2 \text{ for all } \omega \leq 0.1 \text{ rad/s.}$$

Design a compensator to achieve this while reducing the phase margin as little as possible. Estimate the new phase margin. [30%]

Two copies of Fig. 1 are provided on separate sheets. These should be handed in with your answers.

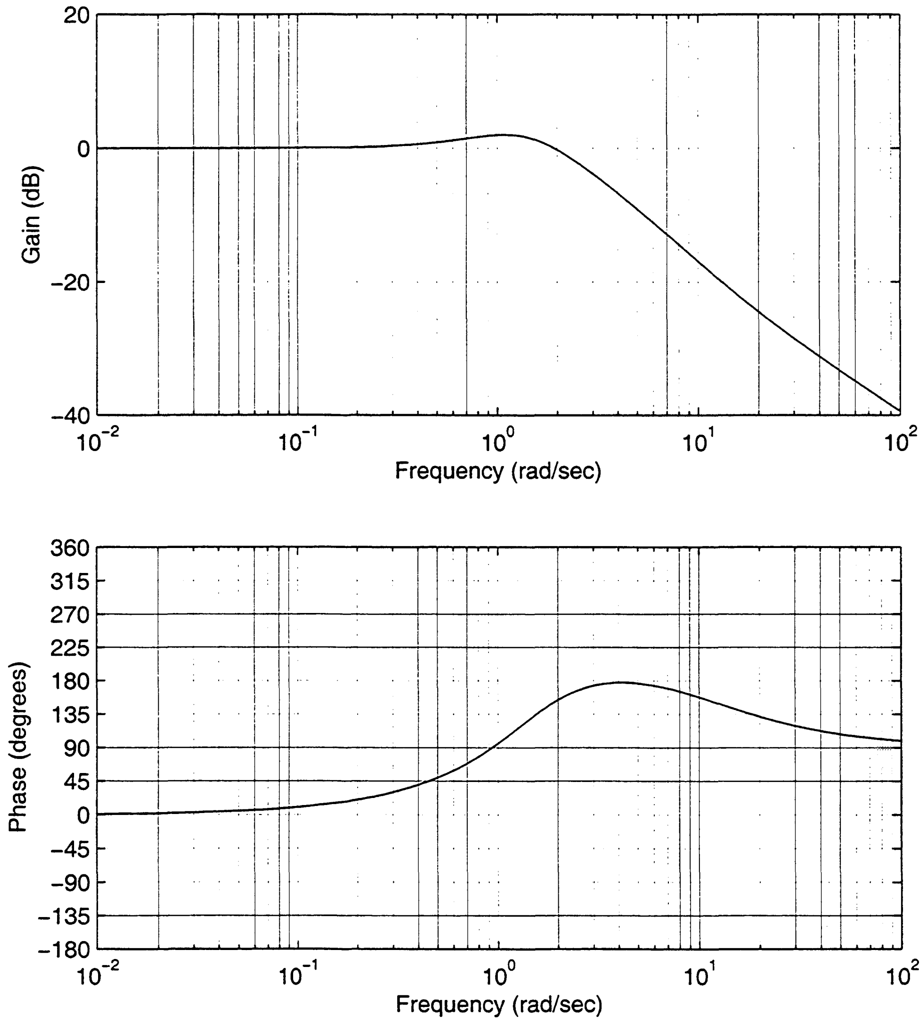


Figure 1: Bode diagram of $G(s)$ for Question 3.

END OF PAPER

UNIVERSITY OF
CAMBRIDGE

UNIVERSITY OF
CAMBRIDGE

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

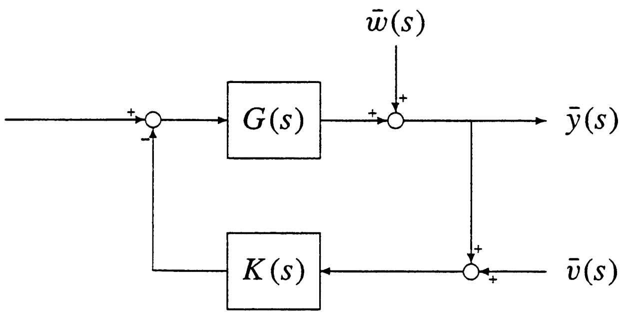
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \qquad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90^\circ.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\mathrm{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^v)| \log\coth\frac{|v|}{2}\, dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

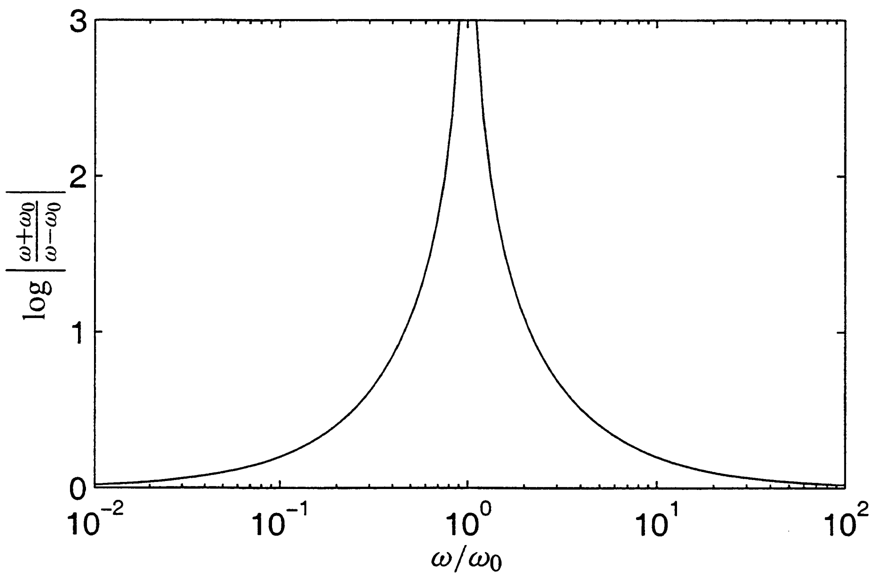


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2}\left.\frac{d\log|L(j\omega_0 e^v|)}{dv}\right|_{\omega=\omega_0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\left(j|s_0|e^{v}\right)\right| dv}$$

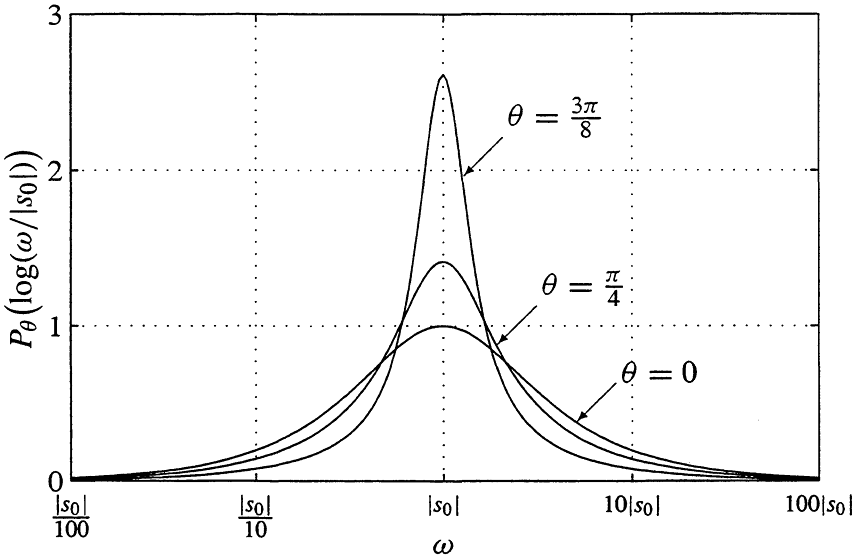
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

G. Vinnicombe
M.C. Smith
November 2002