UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Wednesday 28 April 2004 2.30 to 4

Module 4F2

ROBUST MULTIVARIABLE CONTROL

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 Consider the multivariable feedback system of Fig. 1 which is assumed to be internally stable for $\Delta = 0$.

(a) Show that this system will be internally stable for all stable perturbations, Δ , with $\|\Delta\|_\infty < \epsilon$ if and only if $\|H(s)\|_\infty \leq 1/\epsilon$, where

$$H(s) = \left(I - G(s)K(s)\right)^{-1} G(s)K(s)$$

[The small gain theorem may be stated without proof.] [30%]

(b) Suppose that $H(0) = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

(i) Calculate $\bar{\sigma}\left(H(0)\right)$, the largest singular value of $H(0)$. [10%]

(ii) Assuming that $\|H(s)\|_\infty = \bar{\sigma}\left(H(0)\right)$, demonstrate that $\Delta = \dfrac{1}{20}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ is a perturbation, with the smallest value of $\|\Delta\|_\infty$, that will destabilize the feedback system. What can you do deduce about the closed-loop poles of the system in the presence of this perturbation? [30%]

(iii) It is now required to determine whether the closed-loop transfer function, $T_{d\rightarrow y}$, satisfies $\bar{\sigma}\left(T_{d\rightarrow y}(j\omega)\right) \leq M$ for all ω , a given constant M and for all Δ satisfying $\|\Delta\|_\infty < \epsilon$. Formulate this as a structured singular value test. [30%]

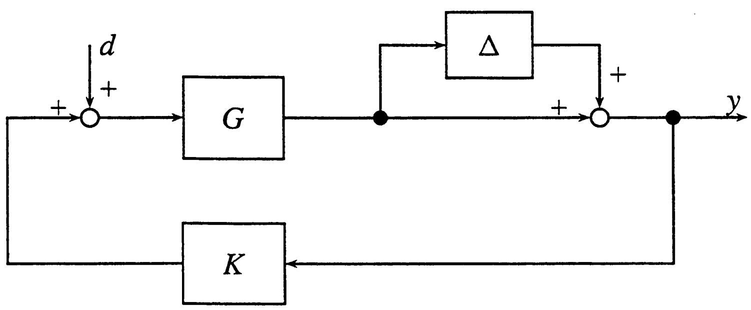

Fig. 1

2 (a) Consider the finite horizon optimal control problem

$$x_{k+1} = f(x_k, u_k),\ x_k \in \mathbb{R}^n,\ u_k \in \mathbb{R}^m$$

$$J\big(x_0, u_0, u_1, \ldots, u_{h-1}\big) = \sum_{k=0}^{h-1} c\big(x_k, u_k\big) + J_h\big(x_h\big)$$

where the sequence $\{u_k : k = 0, 1, \ldots, h-1\}$ is to be chosen to minimize J. Define the "value function" $V(x, k)$ for this problem and explain the significance of $V(x, 0)$ and $V(x, h)$. Recall that $V(\cdot, \cdot)$ solves the Dynamic Programming equation

$$V(x, k) = \min_u\big(c(x, u) + V(f(x, u), k+1)\big)$$

Briefly describe the principle that this equation captures, and explain how it simplifies the problem. [40%]

(b) Consider now the particular optimal control problem with

$$x_{k+1} = x_k + u_k$$

$$J = x_h^2 + \sum_{k=0}^{h-1} u_k^2$$

(i) State the Dynamic Programming equation for this problem. [10%]

(ii) Assuming that V has the form

$$V(x, k) = f(k)\, x^2$$

find the recursion satisfied by $f(k)$. [30%]

(iii) Find the optimal cost if $h = 5$ and $x_0 = 1$. Find also the corresponding sequence of optimal controls. [20%]

3 Consider the closed loop transfer function from $\begin{bmatrix} w \\ v \end{bmatrix}$ to $\begin{bmatrix} y \\ u \end{bmatrix}$ in Fig. 2.

(a) Find the generalized plant $P(s)$ such that

$$\begin{bmatrix} y \\ u \end{bmatrix} = \mathcal{F}_l\big(P(s), K(s)\big) \begin{bmatrix} w \\ v \end{bmatrix}$$

and evaluate $\mathcal{F}_l\big(P(s), K(s)\big)$. [30%]

(b) If $G(s)$ has a state-space realization

$$\dot{x} = ax + ku$$
$$y = x$$

where $a > 0$ and $k > 0$, find a state-space realization of $P(s)$ as defined in (a). [20%]

(c) For $G(s)$ as in (b) find

$$\inf_{K(s) \text{ stabilizing}} \left\| \mathcal{F}_l\big(P(s), K(s)\big) \right\|_\infty$$

and comment on its dependence on a and k. [40%]
[Recall that the CARE and FARE are given by $XA + A^T X + C^T C - XBB^T X = 0$ and $YA^T + AY + BB^T - YC^T CY = 0$ respectively.]

(d) What set of systems is any controller achieving the infimum in (c) guaranteed to stabilize? [10%]

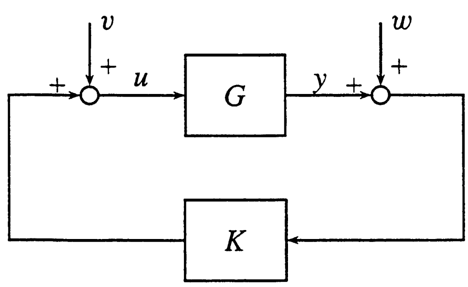

Fig. 2

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