ENGINEERING TRIPOS PART IIB

Tuesday 4 May 2004 2.30 to 4

Module 4F3

NONLINEAR AND PREDICTIVE CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) Outline Lyapunov's *direct* and *indirect* methods for proving the stability of an equilibrium of a system $\dot{x} = f(x)$. [20%]

(b) Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= -x_1 - (\alpha x_1 + \beta x_2)^2 x_2\end{aligned}$$

(i) Find all its equilibrium points. [20%]

(ii) Using the function

$$V(x) = x_1^2 + x_2^2$$

show that this system is globally asymptotically stable if $\beta \neq 0$. [20%]

(iii) Is it asymptotically stable if $\beta = 0$? [20%]

(c) Explain briefly what is meant by the term *hybrid system*, and give an example of such a system. [20%]

2 Consider the nonlinear *dead-zone* characteristic defined by

$$f(e) = \begin{cases} e + \delta & \text{if } e \leq -\delta \\ 0 & \text{if } -\delta < e < \delta \\ e - \delta & \text{if } \delta \leq e \end{cases}$$

(a) If the input to the dead-zone nonlinearity is $e(t) = E\sin(\omega t)$, and $E > \delta$, show that its *describing function* is given by [30%]

$$N(E) = \frac{2}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{\delta}{E}\right) - \left(\frac{\delta}{E}\right)\sqrt{1 - \left(\frac{\delta}{E}\right)^2}\right]$$

(b) What is $N(E)$ if $E < \delta$? [5%]

(c) What is the limiting value of $N(E)$ as $E \to \infty$? Give an intuitive explanation of this value. [5%]

(d) If the dead-zone characteristic is connected in negative feedback with a linear system whose transfer function is

$$G(s) = \frac{k}{s(s+1)^2}$$

for what values of k is a limit cycle oscillation predicted by the describing function method? [25%]

Is this limit cycle stable? [5%]

(e) It can be shown that the transfer function given in (d) has the property that

$$\mathbf{Re}\{G(j\omega)\} > -2k$$

if $k > 0$. Use the *Circle criterion* and/or the *Popov criterion* to specify a range of values of k for which the feedback system defined in (d) is globally asymptotically stable. [30%]

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3 You are given the following discrete-time system with 1 state and 1 input:

$$x(k+1) = 3x(k) + u(k).$$

At each time instant k, a measurement of the state is available and this measurement is denoted by $x(k)$. Let x_s and u_s be the prediction of the state and input, respectively, at time $k+s$, namely $x_0 = x(k)$ and $x_{s+1} = 3x_s + u_s$ for $s = 0, 1, \ldots$. The vectors U and X are defined as

$$U := \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \text{ and } X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

You will be asked below to design a state feedback, receding horizon controller based on the following finite horizon cost:

$$V(x(k), U) := Px_2^2 + \sum_{s=0}^{1} (x_s^2 + u_s^2)$$

where $P > 0$. (Note that P is a scalar.)

(a) Show that

$$X = \Phi x(k) + \Gamma U,$$

where the vector Φ and the matrix Γ are defined as [10%]

$$\Phi := \begin{bmatrix} 3 \\ 9 \end{bmatrix} \quad \text{and} \quad \Gamma := \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

(b) Show that the cost function can be written as

$$V(x(k), U) = U^T \left(I + \Gamma^T \Omega \Gamma\right) U + 2U^T \Gamma^T \Omega \Phi x(k) + \left(1 + \Phi^T \Omega \Phi\right) x(k)^2,$$

where the matrix Ω is defined as [20%]

$$\Omega := \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}.$$

(c) Show that the expression for the optimal input sequence $U^*(x(k))$ which minimises $V(x(k), U)$ is given by [30%]

$$U^*(x(k)) = -(I + \Gamma^T \Omega \Gamma)^{-1} \Gamma^T \Omega \Phi x(k).$$

(d) From (c), obtain an expression for a receding horizon control law. [5%]

(e) State sufficient conditions which allow the terminal cost to be employed as a Lyapunov function to guarantee that the control law found in (d) is stabilising. [15%]

(f) Using (e), show that if $u = Kx$ is the terminal control law, then the receding horizon controller is stabilising if [20%]

$$-4 < K < -2 \quad \text{and} \quad (K^2 + 6K + 8)P \leq -1 - K^2.$$

4 You are given the following discrete-time system with 1 state and 1 input:

$$x(k+1) = 2x(k) + u(k).$$

At each time instant k, a measurement of the state is available and this measurement is denoted by $x(k)$. Let x_s and u_s be the prediction of the state and input, respectively, at time $k+s$, namely $x_0 = x(k)$ and $x_{s+1} = 2x_s + u_s$ for $s = 0, 1, \ldots$. The vectors U and X are defined as

$$U := \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \text{ and } X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Show that

$$X = \Phi x(k) + \Gamma U,$$

where the vector Φ and matrix Γ are defined as [10%]

$$\Phi := \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad \Gamma := \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

(b) The following constraints on the state and input are given:

$$\begin{aligned} -2 &\leq x_1 \leq 2 \\ -1.6 &\leq x_2 \leq 0.8 \\ -1 &\leq u_s \leq 2, \quad s = 0, 1 \end{aligned}$$

Compute the matrices C and E and vectors d and f such that these constraints can be written as [20%]

$$CX \leq d \quad \text{and} \quad EU \leq f.$$

(c) Find an expression for the matrix G and the vectors h and L, in terms of C, Φ, Γ, E, d and f, such that the constraints in (b) can be written as [30%]

$$GU \leq h + Lx(k).$$

(d) Assuming the terminal control law is $u(k) = Kx(k) = -1.25x(k)$, show that the terminal constraint defines a set of states that is input-admissible under this terminal control law. [10%]

(e) Assuming the same terminal control law as in (d), determine whether the terminal constraint defines an invariant set of states for the closed-loop system: [15%]

$$x(k+1) = (2+K)x(k).$$

(f) Assuming that we are interested in regulating the system around the origin, is a receding horizon controller with the above constraints feasible for all time if a feasible input sequence is found at time 0? Give a brief explanation to justify your claim. [15%]

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