UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Monday 26 April 2004 9 to 10.30

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Consider the following signal $\{u(n)\}$ given by

$$u(n) = \alpha u(n-1) + v(n)$$

where $|\alpha| < 1$ and $\{v(n)\}$ is zero-mean white noise of variance $E\left(v^2(n)\right) = \sigma_v^2$.

This signal is filtered through a linear filter having impulse response coefficients $\beta(i)$, $(i = 0, 1, ..., L-1)$ and the observations are

$$y(n) = \sum_{i=0}^{L-1} \beta(i)\, u(n-i) + w(n)$$

where $\{w(n)\}$ is zero-mean white noise of variance $E\left(w^2(n)\right) = \sigma_w^2$. The noise $\{v(n)\}$ is uncorrelated with $\{w(n)\}$.

(a) Consider the vector $\mathbf{u}(n) = (u(n)\ u(n-1)\ \ldots\ u(n-M+1))^{\mathrm{T}}$ where $M > L$. Derive expressions for $E\left(\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right)$ and $E(y(n)\,\mathbf{u}(n))$. [50%]

(b) Give the explicit expression for the Wiener filter $\mathbf{h}_{\text{opt}}$ that minimises

$$J(\mathbf{h}) = E\left(\left(y(n) - \mathbf{h}^{\mathrm{T}}\mathbf{u}(n)\right)^2\right)$$

and compute $J\left(\mathbf{h}_{\text{opt}}\right)$.

Hint. No additional calculation is required. [25%]

(c) In a real-world environment, the filter coefficients $\beta(i)$, $(i = 0, 1, ..., L-1)$ are unknown so the Wiener filter cannot be implemented. Describe a LMS algorithm to approximate the Wiener filter. Would you recommend the use of the LMS algorithm if $|\alpha| \simeq 1$? Explain your answer, giving potential alternatives if you would not recommend LMS. [25%]

2 (a) Consider the following recursive algorithm (whitened gradient search method)

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu \mathbf{R}^{-1} (\mathbf{p} - \mathbf{R}\mathbf{h}(n-1)) \quad (1)$$

where $\mathbf{R}$ and $\mathbf{p}$ are respectively a definite positive matrix and a vector of appropriate dimensions. Assuming $\mathbf{h}(n)$ converges towards a limit $\mathbf{h}_{\text{opt}}$, find an expression for $\mathbf{h}_{\text{opt}}$.

[25%]

(b) Using Eq. (1) and the expression for $\mathbf{h}_{\text{opt}}$, obtain a recursion for

$$\mathbf{v}(n) = \mathbf{h}(n) - \mathbf{h}_{\text{opt}}.$$

Deduce a condition on μ ensuring convergence of the recursive algorithm whatever initial vector $\mathbf{v}(0)$ is chosen. [30%]

(c) Let $\mathbf{R}$ and $\mathbf{p}$ be given by $\mathbf{R} = E\left\{\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right\}$ and $\mathbf{p} = E\{\mathbf{u}(n)\,d(n)\}$ where $\mathbf{u}(n) = (u(n)\ u(n-1)\ \ldots\ u(n-M+1))^{\mathrm{T}}$. In practice, $\mathbf{R}$ and $\mathbf{p}$ are typically unknown and one only has access to a realisation of the input signal $\{u(n)\}$ and of the reference signal $\{d(n)\}$. The signals $\{u(n)\}$ and $\{d(n)\}$ are assumed stationary and ergodic. We consider the following recursive algorithm

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu\,[\mathbf{R}(n)]^{-1}\left(d(n) - \mathbf{u}^{\mathrm{T}}(n)\,\mathbf{h}(n-1)\right)\mathbf{u}(n) \quad (2)$$

where

$$\mathbf{R}(n) = \left(1 - \frac{1}{n}\right)\mathbf{R}(n-1) + \frac{1}{n}\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n). \quad (3)$$

Explain why Eq. (2)-(3) can be interpreted as a stochastic gradient approximation of Eq. (1). [20%]

(d) Assume now that the signal $\{u(n)\}$ is not stationary. Explain why the algorithm defined by Eq. (2)-(3) would not be useful in this context. Suggest a modification that allows for a non-stationary environment. [25%]

(TURN OVER

3 (a) Describe the parametric approach to power spectrum estimation. Your discussion should include the ARMA, AR and MA models and a comparison of parametric methods with non-parametric methods such as the periodogram. [30%]

(b) Show that the autocorrelation function $R_{XX}[k]$ for a Q th order moving average (MA) model can be expressed as

$$R_{XX}[k] = c_k, \quad k = 0, 1, ..., Q$$

where the terms c_k should be carefully expressed in terms of the moving average parameters b_q , $(q = 0, 1, ..., Q)$. [25%]

(c) Three values for the autocorrelation function of a random process are determined as:

$$R_{XX}[0] = 1.00, \quad R_{XX}[1] = -0.48, \quad R_{XX}[2] = 0.19$$

(up to two decimal places).

Fit a minimum phase MA model with $Q = 2$ to this data using the spectral factorisation method, carefully explaining the steps in your working. [45%]

You may use the following factorisation to assist in answering part (c):

$$z^{-2} - 2.5z^{-1} + 5.25 - 2.5z + z^2 =$$
$$z^{-2}(z - 0.5\exp(i\pi/3))(z - 0.5\exp(-i\pi/3))(z - 2\exp(i\pi/3))(z - 2\exp(-i\pi/3))$$

4 (a) Discuss the effects of time-domain windowing in the spectrum analysis of discrete-time signals. You should include a discussion of spectral leakage, spectral smearing and contrast the properties of Hamming, Hanning and rectangular windows. [40%]

(b) The periodogram estimate for the power spectrum estimate of a random process can be expressed as:

$$\hat{S}_X(e^{j\omega T}) = \sum_{k=-(N-1)}^{N-1} \hat{R}_{XX}[k]\, e^{-jk\omega T}$$

where $\hat{R}_{XX}[k]$ is the estimated autocorrelation sequence, given by:

$$\hat{R}_{XX}[k] = \frac{1}{N} \sum_{n=0}^{N-1-k} x_n\, x_{n+k} \qquad 0 \leq k < N$$

and

$$\hat{R}_{XX}[k] = \hat{R}_{XX}[-k], \qquad -N < k < 0$$

Show that $\hat{R}_{XX}[k]$ is a biased estimate of the autocorrelation function, and hence show that the expected value of the periodogram is given by:

$$E[\hat{S}_X(e^{j\omega T})] = \sum_{k=-\infty}^{\infty} w_k R_{XX}[k]\, e^{-jk\omega T}$$

where w_k is the Bartlett window, defined as:

$$w_k = \begin{cases} \frac{N-|k|}{N}, & |k| < N \\ 0, & \text{otherwise} \end{cases}$$

[30%]

(c) The two dominant frequency components in a random process are at frequencies ω_1 rad/s and ω_2 rad/s. The sampling period is $T = 0.1s$. Determine approximately the minimum window length NT required to resolve the two frequency components in the power spectrum estimate, if they are likely to be spaced as little as 0.2π rad/s apart. State clearly any assumptions you make. You may assume that the normalised 3dB bandwidth of the Bartlett window having length M samples is $1.28(2\pi/M)$ radians and its 6dB bandwidth is $1.78(2\pi/M)$ radians. [30%]

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