UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB
ENGINEERING TRIPOS PART IIA

Friday 30 April 2004 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

Candidates may bring their notebooks to the examination.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) A function $u(x, y)$ is governed by the following partial differential equation.

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} \ .$$

(i) Determine the type of the differential equation and find its characteristic equation. Sketch the characteristic lines on the x-y plane. [25%]

(ii) Find the canonical form of the equation and the general solution. [25%]

(b) Find the function $u(x, y)$ that satisfies the following partial differential equation and boundary conditions.

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy}, \qquad x \in (0, \infty), \qquad y \in [0,1],$$

$$u(0, y) = 0,$$

$$u(x, 0) = \beta,$$

$$u(x,1) = \alpha \exp\left(-\frac{1}{x}\right) + \beta \exp\left(-\frac{1}{x^2}\right).$$ [50%]

2 A function $u(x, t)$ satisfies the following one-dimensional wave equation and boundary conditions.

$$u_{tt} - a^2 u_{xx} = f(x,t), \qquad x \in (-\infty, \infty), \qquad t > 0,$$

$$u(x,0) = 0, \qquad u_t(x,0) = 0.$$

(a) Write down the differential equation and boundary conditions satisfied by the Green's function $G(x,t,x_0,t_0)$ associated with this problem. [20%]

(b) Derive an expression for the Green's function $G(x,t,x_0,t_0)$. [20%]

(c) Use the Green's function to derive a solution for u in terms of the forcing function $f(x,t)$. [30%]

(d) Show that the problem is well posed, i.e. the solution $u(x, t)$ found in part (c) is unique and stable in regard to $f(x,t)$. [30%]

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3 A shear deforming beam of length L has a vertical deflection $y(x)$ and a shearing angle $\psi(x)$, where x is the spatial coordinate measured along the axis of the beam. For harmonic vibrations of frequency ω a functional U is defined as

$$U = \int_0^L \left\{ EI\left(\frac{\partial \psi}{\partial x}\right)^2 + k\left(\frac{\partial y}{\partial x} - \psi\right)^2 - \omega^2 m y^2 - \omega^2 I_P \psi^2 \right\} \mathrm{d}x$$

where E, I, k, m, and I_P are constants relating to the physical properties of the beam.

(a) The vibrations of the beam must satisfy the condition $\delta U = 0$. Derive the differential equations that govern $y(x)$ and $\psi(x)$, and list suitable boundary conditions at $x = 0$ and $x = L$. [40%]

(b) By eliminating $\psi(x)$ from the differential equations derived in part (a), show that $y(x)$ must satisfy the following equation

$$EI\frac{\partial^4 y}{\partial x^4} + \omega^2\left(I_P + \frac{mEI}{k}\right)\frac{\partial^2 y}{\partial x^2} - \omega^2 m\left(1 - \frac{\omega^2 I_P}{k}\right)y = 0$$ [30%]

(c) Find a new functional V, that depends upon $y(x)$ alone (i.e. the functional does not involve ψ), such that $\delta V = 0$ leads to the equation given in part (b). [30%]

4 (a) A surface S within a fluid encloses a region V. The fluid potential function ϕ satisfies the Laplace equation $\nabla^2\phi = 0$ in V. Show that

$$\int_S \nabla\phi(\nabla\phi\cdot\mathbf{n})\,\mathrm{d}S = (1/2)\int_S (\nabla\phi\cdot\nabla\phi)\mathbf{n}\,\mathrm{d}S$$ [30%]

(b) The momentum equation for unsteady flow in the fluid described in part (a) can be written in the form

$$\int_S p\mathbf{n}\,\mathrm{d}S = -\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{u}\,\mathrm{d}V$$

where p is the pressure, ρ is the density, and $\mathbf{u}$ is the velocity vector, given by $\mathbf{u} = \nabla\phi$. The total derivative D/Dt has the following property for any vector $\mathbf{y}$

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathbf{y}\,\mathrm{d}V = \int_V \frac{\partial\mathbf{y}}{\partial t}\mathrm{d}V + \int_S \mathbf{y}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S$$

Assuming that the fluid is incompressible, so that ρ is constant, show that the following equations are valid.

(i) $p = -\rho\dfrac{\partial\phi}{\partial t} - \dfrac{\rho}{2}\mathbf{u}\cdot\mathbf{u}$

(ii) $\nabla p = -\rho\dfrac{\partial\mathbf{u}}{\partial t} - \rho(\mathbf{u}\cdot\nabla\mathbf{u})$

(iii) $\dfrac{\mathrm{D}}{\mathrm{D}t}\displaystyle\int_V \rho\,\mathrm{d}V = 0$ [50%]

(c) For the fluid considered in parts (a) and (b), derive an expression for $\nabla^2 p$ in terms of the velocity potential ϕ. [20%]

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