UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB
ENGINEERING TRIPOS PART IIA

Thursday 22 April 2004 9 to 10.30

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

Answer not more than ***three*** *questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4M13 datasheet (4 pages).

Answers to Sections A and B should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

1 (a) Identify the singularities of the complex function

$$\frac{1}{\left(z^2+1\right)^{1/2}\sin z} \quad .$$ [15%]

(b) Calculate the Laurent expansion of the complex function

$$f(z)=\frac{e^{i4z}}{z-\sin z}$$

about the point z=0 up to the constant term of the series. Hence, deduce the residue at z=0. [40%]

(c) Sketch the mappings in the complex plane, and label salient points, for:

(i) $w(z)=\sqrt{z-1}+1$;

(ii) $w(z)=1/z$;

(iii) $w(z)=\dfrac{1}{\sqrt{z-1}+1}$. [45%]

2 (a) Calculate the following integral using contour integration and the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{4x^2 + 2x + 1} dx \ .$$ [40%]

(b) Consider the integral

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}\left(1 + x^2\right)} dx \ .$$

(i) By replacing x with the complex variable z and by considering the contour shown in Figure 1, demonstrate that $I = J/2$ where

$$J = \oint \frac{1}{\sqrt{1 - z^2}\left(1 + z^2\right)} dz \ .$$

(ii) Hence evaluate J. [60%]

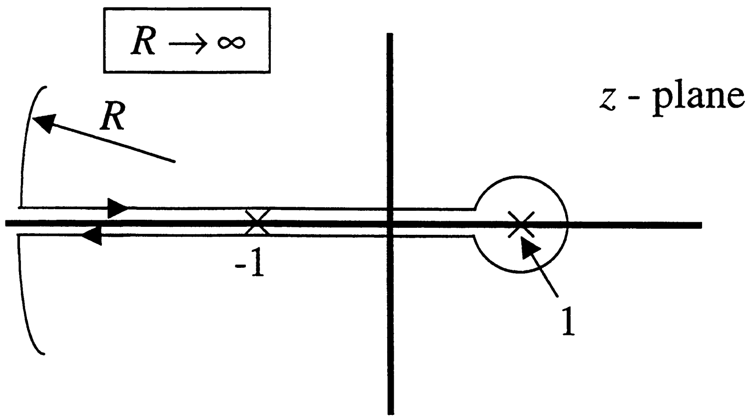

Fig. 1

SECTION B

3 The efficiency η with which a power screw transmits power is, to a good approximation, given by

$$\eta = x\left(\frac{y-\mu x}{xy+\mu}\right)$$

where μ is the coefficient of friction between the screw and the object it is driving, $x = \tan\lambda$, where λ is the lead angle of the screw, and $y = \cos\alpha$, where α is the angle at which the flanks of the screw thread are inclined.

The lead angle λ must be greater than 0°. To allow the use of an adjustable split nut for wear compensation, α must be at least 15°.

Physically both λ and α must be less than 90°, but these bounds do not affect the optimal design of power screws and can therefore be ignored.

(a) Formulate the task of optimizing the efficiency of the screw as a constrained minimization problem with two control variables, x and y. [15%]

(b) By considering the gradient of the objective function, show that there can be no unconstrained minimum for this problem, and that the constraint arising from wear compensation considerations is active at the optimum. [20%]

(c) Estimate, using a Golden Section line search, the value of x (and hence λ) that optimizes the efficiency of the screw for the case $\mu = 0.1$. A suitable initial interval for x is between 0.1 and 2.9, and the search can be halted when the interval has been reduced four times. [40%]

(d) Find analytically the value of x (and hence λ) that optimizes the efficiency of the screw for the case $\mu = 0.1$, and comment on the performance of the Golden Section line search.

How does the optimal value of x vary as μ increases? [25%]

4 The control system for a nuclear submarine's reactor consists of n components, each of which has a failure rate x_i ($i = 1,...,n$) and whose cost is C_i / x_i^2 where each C_i is a known, positive constant. The reliability of the system, operating for a given length of time T, is

$$R(\underline{x}) = \exp\left(-T\sum_{i=1}^{n} x_i\right).$$

An engineer has been asked to design a minimum cost control system satisfying a constraint on the reliability, $R(\underline{x}) \geq R_0$.

(a) Explain why the constraint on the reliability must be active at the optimum. [10%]

(b) Given that the reliability constraint is active, it can be treated as an equality constraint. Use the Lagrange multiplier method to show that the optimal component failure rates are given by

$$x_i = -\frac{\ln(R_0)\sqrt[3]{C_i}}{T\sum_{j=1}^{n}\sqrt[3]{C_j}}$$

You may find it helpful to reformulate the constraint equation by taking logarithms. It is not necessary to check the second-order optimality conditions. [45%]

(c) In a nuclear submarine space is at a premium. The available technologies for the components of the control system are such that the volume of the system is approximately

$$V(\underline{x}) = \sqrt{\sum_{i=1}^{n} V_i x_i^2}$$

If $n = 2$, $T = 10^5$ hr, $V_1 = V_2 = 6.25\times10^{18}$ m^6 hr^2, $C_1 = 1\times10^{-13}$ £ hr^{-2} and $C_2 = 2\times10^{-13}$ £ hr^{-2}, by graphically identifying the feasible region, or otherwise, find the minimum cost control system satisfying the constraints $R(\underline{x}) \geq 0.9999$ and $V(\underline{x}) \leq 2$ m^3. [45%]

END OF PAPER

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4M13

OPTIMIZATION

DATA SHEET

1. Taylor Series Expansion

For one variable:

$$f(x) = f(x^*) + (x - x^*) f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

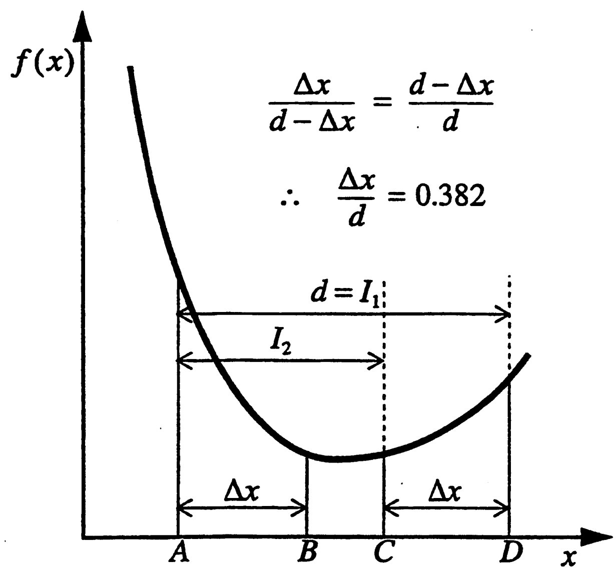
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \boldsymbol{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } \boldsymbol{H}(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$\boldsymbol{H}(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. Golden Section Method



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -H(\mathbf{x}_k)^{-1}\nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T\mathbf{d}_k}{\mathbf{d}_k^T H(\mathbf{x}_k)\mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T\mathbf{d}_0}{\mathbf{d}_0^T H(\mathbf{x}_0)\,\mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|}\right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k\mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T\nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T H(\mathbf{x}_{k+1})\,\mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T\mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\,J(\mathbf{x}_k)^T J(\mathbf{x}_k)\,]^{-1} J(\mathbf{x}_k)^T\mathbf{r}(\mathbf{x}_k)$

$$\text{where } J(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \begin{bmatrix} \nabla h_1(\mathbf{x}^*) & \ldots & \nabla h_m(\mathbf{x}^*) \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \leq 0, i = 1, \ldots, p$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \\ \forall\ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) &= 0 \quad (p \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \geq 0$ are the Kuhn-Tucker multipliers.

9. Penalty & Barrier Functions

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \le 0$, $i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0, g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

$$\text{minimise } q(\mathbf{x}, p_k)$$