Module 4A6 Flow induced sound and vibration
Solutions 2005

1 (a) (i) The pressure is

$$p = \frac{\alpha e^{i\omega t - i\omega r/c}}{r} + \frac{\beta e^{i\omega t + i\omega r/c}}{r}$$

The term α are outgoing waves (ie propagate towards $r=\infty$), while the β term are incoming waves (ie towards $r=0$).

$\therefore$ outgoing waves only $\Rightarrow \beta = 0$

moreover $p(r=a) = Ae^{i\omega t} = \frac{\alpha e^{i\omega t - i\omega a/c}}{a}$

$$\Rightarrow \alpha = aAe^{i\omega a/c}$$

$$\therefore p = A\frac{a}{r}e^{i\omega t - i\omega(r-a)/c}$$ [10%]

(ii) As $r \to 0$ $p \to \frac{e^{i\omega t}}{r}\left\{\alpha e^{-i\omega r/c} + \beta e^{i\omega r/c}\right\}$

The denominator $\to 0$, so for a finite pressure @ $r=0$ require numerator $\to 0$ as well $\Rightarrow \alpha + \beta = 0$ (1).

Still have $p(r=a) = Ae^{i\omega t} \Rightarrow A = \frac{\alpha e^{-i\omega a/c} + \beta e^{i\omega a/c}}{a}$ (2)

(1) and (2) $\Rightarrow \alpha\left(e^{-i\omega a/c} - e^{i\omega a/c}\right) = aA$

$$\Rightarrow \alpha = \frac{-aA}{2i\sin(\omega a/c)}, \quad \beta = \frac{aA}{2i\sin(\omega a/c)}$$

$$\therefore p(r) = \frac{e^{i\omega t}\, aA}{2ir\sin(\omega a/c)}\left\{-e^{-i\omega r/c} + e^{i\omega r/c}\right\}$$

$$= \frac{e^{i\omega t}\, aA \sin(\omega r/c)}{r\sin(\omega a/c)}$$ [15%]

Know that $p = -\rho_0\frac{\partial\phi}{\partial t} = -i\rho_0\omega\phi$

$$\text{so } \phi = \frac{ip}{\rho_0\omega} = \frac{iaA}{\rho_0\omega r}e^{i\omega t}\frac{\sin(\omega r/c)}{\sin(\omega a/c)}$$

Module 4A6 Flow induced sound and vibration
Solutions 2005

(. cont.)

[A]coustic velocity $= \nabla\phi = \frac{\partial\phi}{\partial r}$ in radial direction

$$= \frac{iaA}{\rho_0\omega\sin\left(\frac{\omega a}{c}\right)} e^{i\omega t} \frac{\partial}{\partial r}\left\{\frac{\sin(\omega r/c)}{r}\right\}$$

$$= \frac{iaA}{\rho_0\omega\sin\left(\frac{\omega a}{c}\right)} e^{i\omega t} \left\{\frac{\frac{\omega r}{c}\cos(\omega r/c) - \sin(\omega r/c)}{r^2}\right\}$$

[10%]

[A]coustic power flux, in radial direction, time averaged

$$= \frac{1}{2}\mathrm{Re}\left(p^*\frac{\partial\phi}{\partial r}\right) = \frac{1}{2}\mathrm{Re}\left\{i\rho_0\omega\phi^*\frac{\partial\phi}{\partial r}\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left(i\rho_0\omega\frac{-iaA^*}{\rho_0\omega r}e^{-i\omega t}\frac{\sin(\omega r/c)}{\sin(\omega a/c)}\frac{iaAe^{i\omega t}}{\rho_0\omega\sin\left(\frac{\omega a}{c}\right)}\frac{\left[\frac{\omega r}{c}\cos\left(\frac{\omega r}{c}\right)-\sin\left(\frac{\omega r}{c}\right)\right]}{r^2}\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(i|A|^2\underline{\qquad\qquad}\right)$$

a real number

$\therefore$ flux $=0$

[10%]

Wavelength of sound is $\lambda = \frac{2\pi}{k}$, $\omega/k = c$

$$\lambda = \frac{2\pi c}{\omega} \gg a \Rightarrow \frac{\omega a}{c} \ll 1$$

Inside $r=a$, have $\frac{\omega r}{c} \ll 1$ as well

$$\therefore p = \frac{aAe^{i\omega t}}{r}\frac{\sin(\omega r/c)}{\sin(\omega a/c)} \simeq \frac{aAe^{i\omega t}}{r}\frac{\omega r/c}{\omega a/c}$$

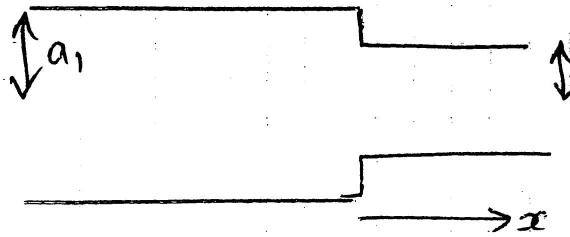
$$= Ae^{i\omega t}$$

$\therefore$ pressure is uniform in $r<a$

[5%]

Qu 1 cont.

(b)



In $x<0$, $\quad p = I e^{i\omega t - i\omega x/c} + R e^{i\omega t + i\omega x/c}$ (1)

In $x>0$ $\quad p = T e^{i\omega t - i\omega x/c}$ (2)

Expect pressure to be continuous across $x=0$

(1) and (2) $\Rightarrow$ $\quad I + R = T$ (3)

$p = -\rho_0 \frac{\partial \phi}{\partial t} = -\rho_0 i\omega \phi$, and speed $= \frac{\partial \phi}{\partial x}$

$$\therefore \quad v = \frac{i}{\rho_0 \omega}\left\{ -\frac{i\omega}{c} I e^{i\omega t - i\omega x/c} + \frac{i\omega}{c} R e^{i\omega t + i\omega x/c} \right\} \quad x<0$$

$$= \frac{i}{\rho_0 \omega}\left\{ -\frac{i\omega}{c} T e^{i\omega t - i\omega x/c} \right\} \quad x>0$$

mass flux in = mass flux out

$$\Rightarrow \quad \rho_0 \pi a_1^2 \, v(x=-0) = \rho_0 \pi a_2^2 \, v(x=+0)$$

$$a_1^2 \left\{ -\frac{\omega I}{c} + \frac{\omega R}{c} \right\} = a_2^2 \left\{ -\frac{\omega T}{c} \right\}$$

$$\frac{a_1^2}{a_2^2}\left(-I + R\right) = -T \qquad (4)$$

(3) + (4) $\Rightarrow$ $\quad I\left(1 - \frac{a_1^2}{a_2^2}\right) + R\left(1 + \frac{a_1^2}{a_2^2}\right) = 0$

$$\therefore \quad R = \frac{I\left(a_1^2 - a_2^2\right)}{a_1^2 + a_2^2}$$

In (3) $\quad T = I + R = I\left\{1 + \frac{(a_1^2 - a_2^2)}{a_1^2 + a_2^2}\right\} = \frac{2a_1^2 I}{a_1^2 + a_2^2}$ [30%]

Qu 1) cont.

time averaged energy flux $= \frac{1}{2}\mathrm{Re}(p^* v)$

reflected $= \frac{1}{2}\mathrm{Re}(p_r^* v_r)$

p_r = reflected pressure $= R e^{i\omega t + i\omega x/c}$

v_r = " speed $= \frac{i}{\rho_0\omega} R \frac{i\omega}{c} e^{i\omega t + i\omega x/c}$

$\therefore$ reflected flux $= \frac{1}{2}\mathrm{Re}\left[R^* R\left(\frac{-\omega}{\rho_0\omega c}\right)\right] = -\frac{1}{2}\mathrm{Re}\left(\frac{R^2}{\rho_0 c}\right)$

(minus sign here, as directed left)

transmitted flux $= \frac{1}{2}\mathrm{Re}(p_t^* v_t)$

$p_r = T e^{i\omega t - i\omega x/c}$

$v_t = \frac{i}{\rho_0\omega} T\left(-\frac{i\omega}{c}\right) e^{i\omega t - i\omega x/c}$

$\therefore$ transmitted flux $= \frac{1}{2}\mathrm{Re}\left(\frac{T^2}{\rho_0 c}\right)$

Require total reflected energy = (flux)(area)

$= \frac{R^2}{\rho_0 c}\pi a_1^2$

total transmitted energy $= \frac{T^2}{\rho_0 c}\pi a_2^2$

$\therefore R^2 a_1^2 = T^2 a_2^2$

$(a_1^2 - a_2^2)^2 a_1^2 = 4a_1^4 a_2^2 \rightarrow (a_1^2 - a_2^2)^2 = 4a_1^2 a_2^2$

$\rightarrow a_1^2 - a_2^2 = \pm 2a_1 a_2$

$a_2^2 \pm 2a_1 a_2 - a_1^2 = 0$

choose positive root in each case, as negative radius not allowed.

either $a_2 = \frac{-a_1 + \sqrt{a_1^2 + a_1^2}}{2} = \left(\frac{\sqrt{2}-1}{2}\right) a_1$ [20%]

or $a_2 = \frac{a_1 + \sqrt{a_1^2 + a_1^2}}{2} = \left(\frac{\sqrt{2}+1}{2}\right) a_1$

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Qu2. a) From the data card

$$p'(\underset{\sim}{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{T_{ij}(\underset{\sim}{y}, t - |\underset{\sim}{x} - \underset{\sim}{y}|/c)\, d^3\underset{\sim}{y}}{4\pi |\underset{\sim}{x} - \underset{\sim}{y}|}$$

where $T_{ij} = \rho v_i v_j + (p' - c^2\rho')\delta_{ij} - \tau_{ij}$.

For $\underset{\sim}{x}$ in the far-field,
$$\frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|} = \frac{1}{|\underset{\sim}{x}|}$$

$$\frac{\partial}{\partial x_i} \sim -\frac{x_i}{|\underset{\sim}{x}|c}\frac{\partial}{\partial t}.$$

In a jet of diameter D, mean velocity U

$T_{ij} \sim \rho_0 U^2$

frequency $\omega \sim \frac{U}{D}$

$\frac{\omega D}{c} \sim \frac{U}{D}\frac{D}{c}$ = Mach number and the jet is compact if the Mach number is low.

Hence, for $\underset{\sim}{x}$ in the far-field,

$$p'(\underset{\sim}{x},t) = \frac{x_i x_j}{|\underset{\sim}{x}|^3 c^2}\frac{\partial^2}{\partial t^2}\int T_{ij}\left(\underset{\sim}{y}, t - \frac{|\underset{\sim}{x}|}{c}\right) d^3\underset{\sim}{y}$$

This gives $p'(\underset{\sim}{x},t) \sim \frac{1}{|\underset{\sim}{x}|c^2}\frac{U^2}{D^2}\rho_0 U^2 D^3 = \frac{D}{|\underset{\sim}{x}|}\rho_0\frac{U^4}{c^2}$

The mean square pressure fluctuation therefore scales on

$$\underline{\underline{\rho_0^2 \frac{U^8}{c^4}\frac{D^2}{|\underset{\sim}{x}|^2}}}$$ which is Lighthill's eighth-power law

b) To reduce by 25 dB need $\frac{U_2^8 D_2^2}{U_1^8 D_1^2} = 10^{-2.5}$ (1)

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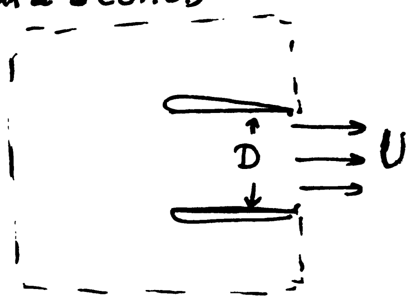
Qu 2 b cont.)



Momentum analysis for large control surface

Air is entrained at rate $\dot{m}_a = \frac{\pi}{4} D^2 U$ with negligible momentum.

Pressure is atmospheric around control surface.

Rate of change of momentum $= \dot{m}_a U =$ Thrust

$$= \frac{\pi}{4} D^2 U^2$$

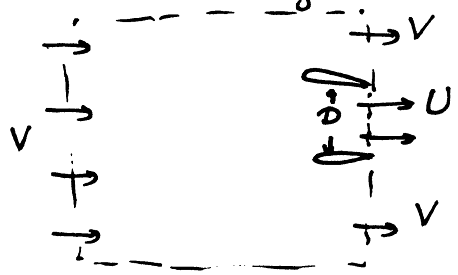
Hence for same thrust $\quad U_2^2 D_2^2 = U_1^2 D_1^2$

$$U_2 = U_1 \frac{D_1}{D_2}$$

Substituting for U_2 in (1) $\quad \frac{D_1^6}{D_2^6} = 10^{-2.5}$

Ratio $\quad \frac{A_2}{A_1} = \left(\frac{D_2}{D_1}\right)^2 = 10^{-2.5/3} = \underline{0.15}$

c) When the aircraft is in forward flight with velocity V, the source is best analysed in a frame of reference stationary relative to the aircraft



Then momentum analysis gives

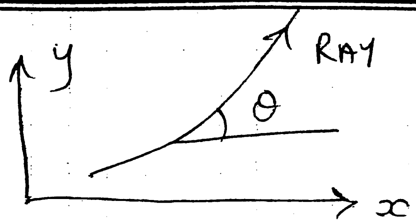
Thrust $= \dot{m}_a (U-V) = \frac{\pi}{4} D^2 U (U-V)$

Hence (i) for same thrust at reduced U need to increase D by more than in the stationary case.

(ii) T_{ij} scales on the turbulent fluctuations which are reduced as U is reduced to be closer to V because the shear flow is reduced.

(iii) Since the aircraft is moving, the sound heard on the ground will undergo a Doppler shift affecting both the frequency heard by a stationary observer and possibly also the amplitude.

3 (a) (i)



Snell's Law is that $\dfrac{\sin\theta}{c(x)}$ = constant on the ray. Ray launched from $x=0$ at angle θ_0

$$\frac{\sin\theta}{x+\beta} = \frac{\sin\theta_0}{\beta} \qquad (*)$$

$$\tan\theta = \frac{dy}{dx} \Rightarrow \sin\theta = \frac{\frac{dy}{dx}}{\left[1+(dy/dx)^2\right]^{1/2}}$$

So back in $(*)$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x+\beta)^2\sin^2\theta_0}{\beta^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x+\beta)^2\sin^2\theta_0}{\beta^2}\Big/\left[1-\frac{(x+\beta)^2\sin^2\theta_0}{\beta^2}\right]$$

$$\frac{dy}{dx} = \frac{(x+\beta)\sin\theta_0}{\beta}\Big/\left[1-\frac{(x+\beta)^2\sin^2\theta_0}{\beta^2}\right]^{1/2}$$

$$\int dy = \frac{\sin\theta_0}{\beta}\int\frac{(x+\beta)}{\left[1-\frac{(x+\beta)^2}{\beta^2}\sin^2\theta_0\right]^{1/2}}\,dx$$

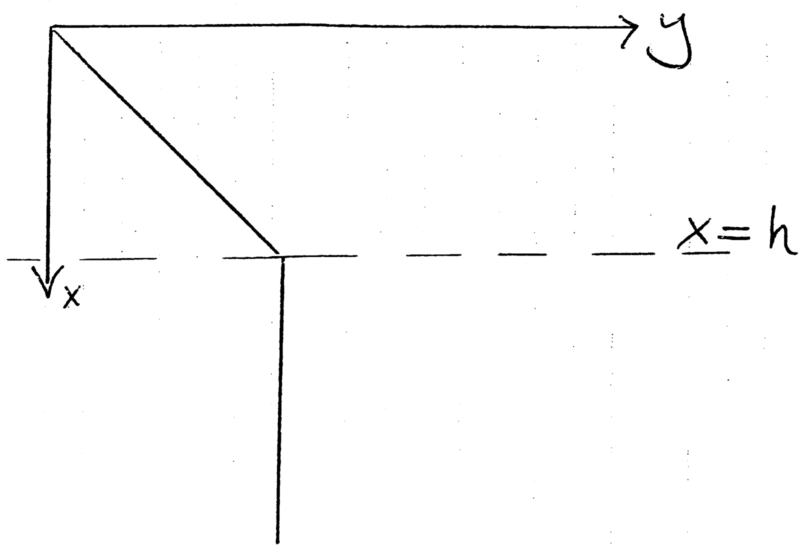
$$y = -\frac{\beta}{\sin\theta_0}\left[1-\frac{(x+\beta)^2}{\beta^2}\sin^2\theta_0\right]^{1/2} + \text{constant}$$

when $x=0$ $y=0$ as ray launched from $(0,0)$

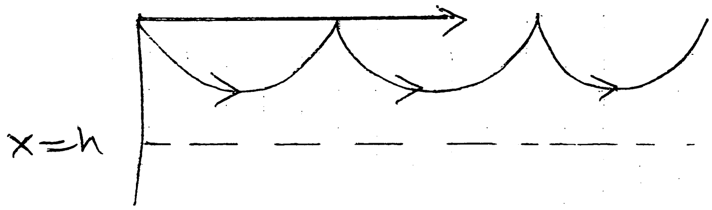
$$y = -\frac{\beta}{\sin\theta_0}\left[1-\frac{(x+\beta)^2}{\beta^2}\sin^2\theta_0\right]^{1/2} + \frac{\beta}{\sin\theta_0}\cos\theta_0$$

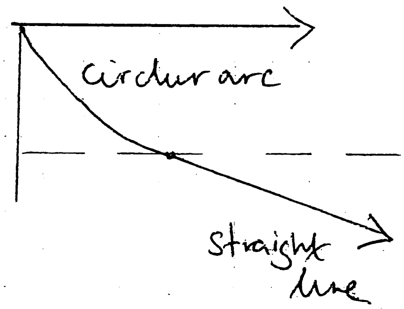
$$\therefore\ \underline{(x+\beta)^2 + (y-\beta\cot\theta_0)^2 = \beta^2\operatorname{cosec}^2\theta_0} \qquad [25\%]$$

A CIRCLE

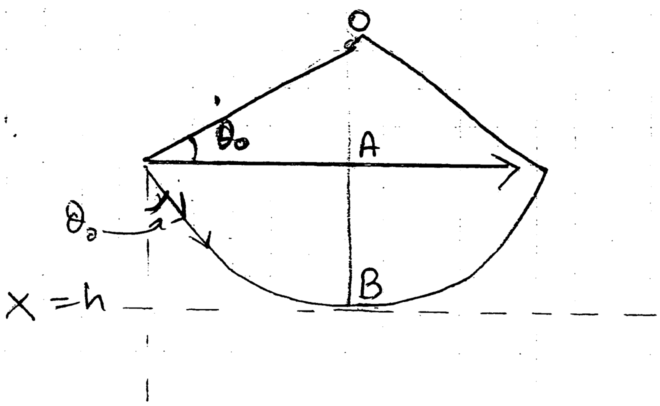


Typical ray trajectories either bend back before reaching $x=h$, and then undergo repeated reflection at free surface, or penetrate into $x>h$ and then become straight





Dividing case:



The radius of the circle is $\beta \operatorname{cosec}\theta_0$

$$|OA| = (\beta \operatorname{cosec}\theta_0) \cdot \sin\theta_0 \qquad |AB| = h$$

$$|OB| = \beta \operatorname{cosec}\theta_0 \quad = |OA| + |AB| \quad = \beta + h$$

$$\therefore \sin\theta_0 = \beta/\beta+h$$

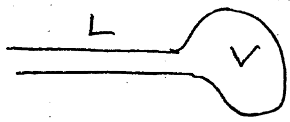
Hence, to return to surface

$$|\theta_0| \geq \sin^{-1}(\beta/\beta+h)$$

[25%]

Qu 3 cont.)

(b)



(i) The volume is V, and $\therefore$ the rate of change of the mass inside the vessel is

$$\frac{\partial \rho}{\partial t} V$$

Single frequency, $\rho' \propto e^{i\omega t}$ where ρ' is the density fluctuation

$\therefore$ rate of change of mass in vessel $= i\omega\rho' V$

$= Q$, since mass only changes by escaping along neck. [10%]

(ii) pressure fluctuation and density fluctuation related by

$$p_2' = c_0^2 \rho'$$

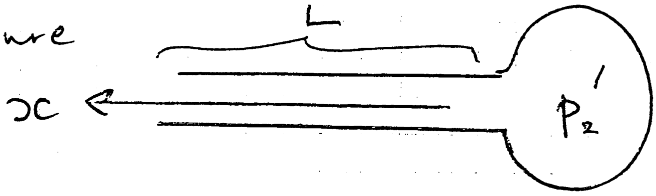
$$= \frac{c_0^2 Q}{iV\omega} \quad \text{from (i)}$$

[10%]

(iii) The one-dimensional Euler equation is

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}$$

p_1' = pressure fluctuation outside



The pressure gradient is $\dfrac{p_1' - p_2'}{L}$

The speed along the pipe is $\propto e^{i\omega t}$

$$\therefore \rho \frac{\partial u}{\partial t} = \rho_0 i\omega u$$

, where ρ_0 = mean density

Qu3 cont.) However, also see that

$$Q = -S\rho_0 u \qquad S = \text{area of pipe cross-section}$$

− sign since u in positive x dir^n $\Rightarrow Q<0$

$$\therefore -\rho_0 i\omega\left(\frac{Q}{S\rho_0}\right) = -\left(\frac{p_1' - p_2'}{L}\right)$$

$$\Rightarrow \frac{p_2' - p_1'}{L} = -\frac{i\omega Q}{S}$$

[20%]

Now use answer from (ii) $\quad p_2' = \dfrac{c_0^2 Q}{iV\omega}$

$$\text{(iii)} \Rightarrow Q\left\{\frac{c_0^2}{iV\omega L} + \frac{i\omega}{S}\right\} = \frac{p_1'}{L}$$

$$p_1' = \frac{Q}{iV\omega L}\left\{c_0^2 - \frac{\omega^2 VL}{S}\right\}$$

$\therefore$ Q is large when p_1' is small if

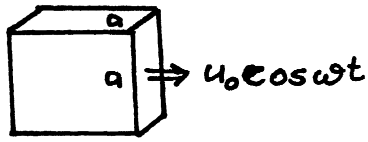
$$c_0^2 - \frac{\omega^2 VL}{S} = 0$$

ie' resonant frequency $c_0\sqrt{\dfrac{S}{VL}}$.

[10%]

Module 4A6 Solutions 2005

Qu 4 a) (i)



This is a monopole source. Since it is compact

$$p'(\underset{\sim}{x},t) = \frac{\rho_0}{4\pi|\underset{\sim}{x}|}\ddot{V}(t-|\underset{\sim}{x}|/c) \quad \text{where } V(t) \text{ is the volume}$$

$$\dot{V}(t) = a^2 u_0 \cos\omega t$$

$$\ddot{V}(t) = -a^2 u_0 \omega \sin\omega t$$

Hence $p'(\underset{\sim}{x},t) = -\dfrac{\rho_0 a^2 u_0 \omega \sin\omega(t-|\underset{\sim}{x}|/c)}{4\pi|\underset{\sim}{x}|}$

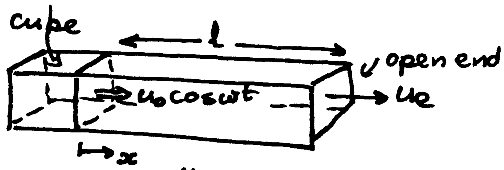
$$\text{Power} = \int_S \frac{\overline{p'^2}}{\rho_0 c}\,dS = \frac{1}{\rho_0 c}\,4\pi|\underset{\sim}{x}|^2\left(\frac{\rho_0 u_0 \omega a^2}{4\pi|\underset{\sim}{x}|}\right)^2\overline{\sin^2\omega t}$$

$$= \rho_0 \frac{a^4}{c}\frac{\omega^2 u_0^2}{4\pi}\frac{1}{2}$$

$$= \underline{\underline{\rho_0 c\left(\frac{\omega a}{c}\right)^2\frac{u_0^2 a^2}{8\pi}}}$$

[35%]

(ii) For cube in duct



The same analysis as above gives

$$\text{Power} = \rho_0 c\left(\frac{a}{c}\right)^2\frac{a^2}{4\pi}\overline{\left(\frac{\partial u_e}{\partial t}\right)^2}$$

where u_e is the velocity at the open end of the duct.

Now need to use duct analysis for plane waves to relate u_e to u_0.

Within the duct, $p'(x,t) = Ae^{i\omega(t-x/c)} + Be^{i\omega(t+x/c)}$

Open end boundary condition,

$$p'(l,t) = 0 \Rightarrow Ae^{-i\omega l/c} + Be^{i\omega l/c} = 0$$

$$B = -Ae^{-2i\omega l/c}$$

$$p'(x,t) = Ae^{i\omega t}\left(e^{-i\omega x/c} - e^{i\omega(x-2l)/c}\right)$$

$$u'(x,t) = \frac{1}{\rho_0 c}Ae^{i\omega t}\left(e^{-i\omega x/c} + e^{i\omega(x-2l)/c}\right)$$

On moving face $x=0$, $u = u_0 e^{i\omega t}$

$$\Rightarrow u_0 = \frac{A}{\rho_0 c}\left(1 + e^{-2i\omega l/c}\right) \Rightarrow A = \frac{\rho_0 c u_0}{1+e^{-2i\omega l/c}}$$

Qu 4 cont.)

$$u_e(t) = u(l,t) = \frac{1}{\rho_0 c} A e^{i\omega t} 2 e^{-i\omega l/c}$$

$$u_e(t) = u_0 e^{i\omega t} \frac{2 e^{-i\omega l/c}}{1 + e^{-2i\omega l/c}} = u_0 e^{i\omega t} \frac{2}{e^{i\omega l/c} + e^{-i\omega l/c}}$$

$$= \frac{u_0 e^{i\omega t}}{\cos(\omega l/c)}$$

Hence $\overline{\left(\frac{\partial u_e}{\partial t}\right)^2} = \frac{u_0^2}{2} \frac{\omega^2}{\cos^2(\omega l/c)}$

Radiated sound power $= \underline{\underline{\rho_0 c \left(\frac{\omega a}{c}\right)^2 \frac{u_0^2 a^2}{8\pi} \frac{1}{\cos^2(\omega l/c)}}}$. [35%]

This expression is valid if the radiated sound power is small compared with the reflected power, i.e. away from the resonance condition $\cos(\omega l/c) = 0$.

b) (i) Now there is no net mass flux $\int u\,dS = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} u_0 \sin\left(\frac{\pi x_2}{a}\right) dx_2\, dx_3$

$$= a u_0 \left[-\frac{a}{\pi}\cos\left(\frac{\pi x_2}{a}\right)\right]_{-a/2}^{a/2} = 0 .$$

However there is a moment of mass flux $\int x_2 u\, dS \neq 0$.

The sound will be dipole, axis in the 2-direction. [15%]

(ii) The waves in the duct are now evanescent in the duct and decay exponentially. u_e is therefore very much smaller than u_0 and negligible energy is radiated. [15%]
To see that the wave are evanescent either note that $k_2 = \pi/a$ leads to a wave speed in the x_2-direction of $\omega/k_2 = \frac{\omega a}{\pi} \ll c$. This subsonic surface speed means waves that decay exponentially in the axial direction. Alternatively, look for a solution of the wave equation of the form $e^{i\omega t} f(x_1)\sin\left(\frac{\pi x_2}{a}\right)$. Substituting into the wave equation gives $\frac{d^2 f}{dx_1^2} + \left(\frac{\pi^2}{a^2} - \frac{\omega^2}{c^2}\right) f = 0$

with solutions $f(x_1) = A\, e^{\left(\pi^2/a^2 - \omega^2/c^2\right)^{1/2} x_1} + B\, e^{-\left(\pi^2/a^2 - \omega^2/c^2\right)^{1/2} x_1}$

ie evanescent.

Module 4A6 FLOW INDUCED SOUND AND VIBRATION DATA CARD

Sources

Point sources

monopole of strength $Q(t)$ at the origin generates a pressure field

$$p'(\mathbf{x},t) = \frac{Q(t-|\mathbf{x}|/c)}{4\pi|\mathbf{x}|}.$$

dipole of strength $\mathbf{F}(t)$ at the origin generates a pressure field

$$p'(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\left[\frac{F_i(t-|\mathbf{x}|/c)}{4\pi|\mathbf{x}|}\right] = \frac{x_i}{4\pi}\left[\frac{1}{|\mathbf{x}|^3}F_i(t-|\mathbf{x}|/c) + \frac{1}{|\mathbf{x}|^2 c}\frac{\partial F_i}{\partial t}(t-|\mathbf{x}|/c)\right]$$

Distributed sources

Monopole, strength $q(\mathbf{x},t)$, wave equation $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)p' = q$, pressure field $p'(\mathbf{x},t) = \int \frac{q(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c)}{4\pi|\mathbf{x}-\mathbf{y}|}d^3\mathbf{y}$

Dipole, strength $\mathbf{f}(\mathbf{x},t)$, wave equation $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)p' = -\nabla\cdot\mathbf{f}$, $p'(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\int \frac{f_i(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c)}{4\pi|\mathbf{x}-\mathbf{y}|}d^3\mathbf{y}$.

Quadrupole, strength $T_{ij}(\mathbf{x},t)$, wave equation $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)p' = \frac{\partial T_{ij}}{\partial x_i \partial x_j}$, $p'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\int \frac{T_{ij}(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c)}{4\pi|\mathbf{x}-\mathbf{y}|}d^3\mathbf{y}$.

Far-field form $|\mathbf{x}| >> |\mathbf{y}|$, $\mathbf{y}$ near origin

$$|\mathbf{x}-\mathbf{y}| \sim |\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + O(|\mathbf{x}|^{-2})$$

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} \sim \frac{1}{|\mathbf{x}|} + O(|\mathbf{x}|^{-2})$$

$$\frac{\partial}{\partial x_i} \sim -\frac{x_i}{|\mathbf{x}|c}\frac{\partial}{\partial t} + O(|\mathbf{x}|^{-1}).$$

Physical sources

Heat addition at a rate $w(\mathbf{x},t)$/unit volume is equivalent to a monopole source of strength $\frac{(\gamma-1)}{c^2}\frac{\partial w}{\partial t}$.

Lighthill's acoustic analogy shows that jet noise is generated by quadrupoles of strength
$T_{ij} = \rho v_i v_j + (p' - c^2\rho')\delta_{ij} - \tau_{ij}$.

The Ffowcs-Williams Hawkings equation shows that foreign bodies in linear motion generate far-field sound

$$p'(\mathbf{x},t) = \frac{1}{4\pi|\mathbf{x}|}\frac{\partial}{\partial t}\int_S \rho_0 \mathbf{n}\cdot\mathbf{u}\left(\mathbf{y},t-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|c}\right)dS + \frac{x_i}{4\pi|\mathbf{x}|^2 c}\frac{\partial}{\partial t}\int n_i p\left(\mathbf{y},t-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|c}\right)dS$$

USEFUL MATHEMATICAL FORMULAE

In spherical polar coordinates (r,θ,ϕ)

$$\nabla p' = \left(\frac{\partial p'}{\partial r}, \frac{1}{r}\frac{\partial p'}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p'}{\partial \phi}\right)$$

For $\mathbf{v} = (v_r, v_\theta, v_\phi)$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\nabla^2 p' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p'}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p'}{\partial\phi^2}.$$

Heaviside function $H(t-\tau) = 1 \;\; t>\tau \;;\; =0 \;\; t<\tau$

δ-functions

Kronecker delta $\delta_{ij} = 1$ if $i=j$; 0 if $i \neq j$

1D δ-function: $\delta(t) = 0$ for $t \neq 0$; $f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$

$\int_{-\infty}^{\infty}\delta(t)dt = 1$ and $\int_{-\infty}^{\infty}\delta(t-\tau)f(t)dt = f(\tau)$ for any function $f(t)$.

3D δ-function: $\delta(\mathbf{x}) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)$;
$\delta(\mathbf{x}) = 0$ for $|\mathbf{x}| \neq 0$;
$f(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) = f(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})$
$\int_V \delta(\mathbf{x})dV = 1$ for any volume V that includes the origin

and

$\int_V \delta(\mathbf{x}-\mathbf{y})f(\mathbf{x})d^3\mathbf{y} = f(\mathbf{x})$ for any function $f(\mathbf{x})$ and volume V that includes $\mathbf{x}$.

$$\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = -4\pi\delta(\mathbf{x}).$$

Autocorrelation

$F(\xi)$, the autocorrelation of $f(y) = \overline{f(y)f(y+\xi)}$

$F(0) = \overline{f^2}$

Integral lengthscale ℓ $\quad \ell\overline{f^2} = \int_{-\infty}^{\infty}F(\xi)d\xi$.

Module 4A6 FLOW INDUCED SOUND AND VIBRATION DATA CARD

USEFUL DATA AND DEFINITIONS

Physical Properties

Speed of sound in an ideal gas $\sqrt{\gamma RT}$, where T is temperature in Kelvins
Speed of sound in sea water, c, is a function of temperature, T

T °C	-4	0	5	10	15	20	25	30
c ms^{-1}	1430.2	1449.5	1471.1	1490.2	1507.1	1521.9	1543.7	1545.9

Units of sound measurement

SPL (sound pressure level) $= 20\log_{10}\left(\dfrac{p'_{rms}}{2.10^{-5}\,\text{Nm}^{-2}}\right)$ dB

IL (intensity level) $= 10\log_{10}\left(\dfrac{\text{intensity}}{10^{-12}\,\text{watts m}^{-2}}\right)$ dB

PWL (power level) $= 10\log_{10}\left(\dfrac{\text{sound power output}}{10^{-12}\,\text{watts}}\right)$ dB

Definitions

Surface impedance , Z_s, relates the pressure perturbation applied to a surface, p' , to its normal velocity v_n; $p' = Z_s v_n$.

Characteristic impedance of a fluid $\rho_0 c$

Nondimensional surface impedance of a surface $Z_s/\rho_0 c$

Transmission loss $= 10\log_{10}\left(\dfrac{\text{incident sound power}}{\text{transmitted sound power}}\right)$

Absorption coefficient of a sound absorber $= \dfrac{\text{sound power absorbed}}{\text{incident sound power}}$

Sound absorption (in metric sabins) $= \sum \alpha_i S_i$ where S_i is surface area (in metres^2) with absorption coefficient α_i .

Reverberation time of a room = time taken for the sound intensity level in the room to drop from 60dB to the threshold of hearing.

Wavelength, λ, for sound waves with angular frequency ω, $\lambda = 2\pi c/\omega$

Wave-number $k = 2\pi/\lambda$

Phase speed $= \omega/k$

Group velocity $= \dfrac{\partial \omega}{\partial k}$

Helmholtz number (or compactness ratio) $= kD$ where D is a typical dimension of the source.

Strouhal number $= \omega D/(2\pi U)$ for sound of frequency ω produced in a flow with speed U, length scale D.

BASIC EQUATIONS FOR LINEAR ACOUSTICS

Conservation of mass $\dfrac{\partial \rho}{\partial t} + \rho_0 \nabla\cdot \mathbf{v} = 0$

Conservation of momentum $\rho_0 \dfrac{\partial \mathbf{v}}{\partial t} + \nabla p' = 0$

Isentropic $c^2 = \left.\dfrac{dp}{d\rho}\right|_s$

These equations combine to give the wave equation $\dfrac{1}{c^2}\dfrac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$

Energy density $e = \frac{1}{2}\rho_0 v^2 + \frac{1}{2} p'^2/\rho_0 c^2$

Intensity $\mathbf{I} = p'\mathbf{v}$

$\text{div}\,\bar{\mathbf{I}} = 0$ for statistically stationary (in time) sound fields.

Velocity potential $\phi(\mathbf{x},t)$ satisfies the wave equation and $p' = -\rho_0 \dfrac{\partial \phi}{\partial t}$, $\mathbf{v} = \nabla\phi$.

SIMPLE WAVE FIELDS

1D or plane wave

The general solution of the 1D wave equation is

$$p'(x_1,t) = f(x_1 - ct) + g(x_1 + ct)$$

where f and g are arbitrary functions.

In a plane wave propagating to the right $p' = \rho_0 c u$. In a plane wave propagating to the left $p' = -\rho_0 c u$, u being the particle velocity.

Spherically symmetric sound fields

The general spherically symmetric solution of the 3D wave equation is

$$p'(r,t) = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}$$

where $r = |\mathbf{x}|$; f and g are arbitrary functions.

cos θ dependence

The general solution of the 3D wave equation with $\cos\theta$ dependence is

$$p'(\mathbf{x},t) = \frac{\partial}{\partial x_1}\left[\frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}\right] = \cos\theta\frac{\partial}{\partial r}\left[\frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}\right]$$