4A8 Environmental Fluid Mechanics

1 (a) Hydrostatic balance : $\frac{dp}{dz} = -\rho g$ (1) (z : +ve upwards)

Adiabatic, isentropic process of perfect gas $\Rightarrow$ $T ds = dh - v dp$

$\Leftrightarrow$ $dh = v dp$

$\Leftrightarrow$ $c_p dT = dp/\rho$

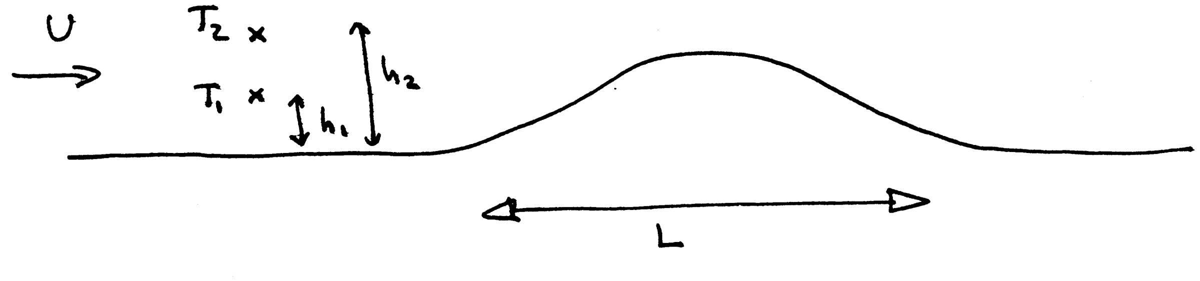
$\Leftrightarrow$ $dp = \rho c_p dT$ (2)

Hence $\frac{dp}{dz} = \rho c_p \frac{dT}{dz} \Rightarrow -\rho g = \rho c_p \frac{dT}{dz}$

$\Leftrightarrow$ $\boxed{\frac{dT}{dz} = -\frac{g}{c_p}}$ QED. (20%)

(b) The Brunt-Vaisala frequency is the natural frequency of oscillation of a parcel of fluid displaced in a density gradient. It comes about due to the fact that a restoring force is applied on the parcel as it is displaced in a stable gradient. (10%)

(c)



(i) Internal wave generation will be a maximum when the driving frequency is equal to the natural frequency of the layer. The driving frequency due to the hill of length L in a wind of velocity U is $\frac{U}{L}$.

The Brunt-Vaisala frequency is

$$N^2 = \frac{g}{T}\left(\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{DALR}\right) \qquad \text{(from Data Card)}$$

$$\frac{dT}{dz} = \frac{T_2 - T_1}{h_2 - h_1}$$; T in above expression is the average temperature $\frac{T_1 + T_2}{2}$;

$$\left.\frac{dT}{dz}\right|_{DALR} = -\frac{g}{c_p}$$

$$\Rightarrow \quad \frac{U^2}{L^2} = \frac{2g}{T_1 + T_2}\left[\frac{T_2 - T_1}{h_2 - h_1} + \frac{g}{c_p}\right]$$

$$\Rightarrow \quad U = L\sqrt{\frac{2g}{T_1+T_2}\left(\frac{T_2 - T_1}{h_2 - h_1} + \frac{g}{c_p}\right)}$$

(50%)

(ii) Logarithmic profile $\Rightarrow \frac{du}{dz} = \frac{u^*}{kz}$

$$Ri = \frac{g}{T}\left[\frac{\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{DALR}}{\left(\frac{du}{dz}\right)^2}\right] \qquad \text{(Data card)}$$

Hence $Ri \geq \frac{1}{4} \Leftrightarrow \frac{2g}{T_1 + T_2}\left[\frac{\frac{T_2 - T_1}{h_2 - h_1} + \frac{g}{c_p}}{\frac{u^{*2}}{k^2 z^2}}\right] \geq \frac{1}{4}$

$$\Leftrightarrow \quad z^2 \geq \frac{T_1 + T_2}{8g} \, \frac{u^{*2}}{\left(\frac{T_2 - T_1}{h_2 - h_1} + \frac{g}{c_p}\right)k^2}$$

$$\Leftrightarrow \quad z \geqslant \frac{u^*}{\kappa}\sqrt{\frac{T_1+T_2}{8g}\,\frac{1}{\left(\frac{T_2-T_1}{h_2-h_1}+\frac{g}{c_p}\right)}}$$

With $u^* = \alpha U$,

$$z \geqslant \frac{\alpha U}{\kappa}\sqrt{\frac{T_1+T_2}{8g}\,\frac{1}{\frac{T_2-T_1}{h_2-h_1}+\frac{g}{c_p}}}$$ is the height that gives $Ri \geqslant 1/4$

When $Ri > \frac{1}{4}$ all turbulence is suppressed. (20%)

2. (a) smallest motion is Kolmogorov scale

$\Rightarrow \eta_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ (from Data Card)

$\Rightarrow \varepsilon = \nu^3/\eta_K^4$

If $\eta_K = 0.1$ mm $\rightarrow$ $\varepsilon = 0.01$ W/kg $(\nu_{water} \sim 10^{-6}$ m^2/s$)$

If we ignore mechanical losses and the kinetic energy of the particles, then

$$\left.\begin{aligned} \text{Power in} &= m \cdot \varepsilon \\ m = \rho \cdot \frac{\pi D^2}{4} \cdot h &= \rho \frac{\pi}{4} \end{aligned}\right\} \Rightarrow \text{Power} = 7.85 \text{ W}$$

(30%)

(b) Using $\varepsilon \sim \frac{u^3}{L} \iff u^3 = \varepsilon \cdot L$

$\iff u = 0.215$ m/s

Hence kinetic energy per unit mass of fluid is $u^2 \Rightarrow 0.0464$ m^2/s^2.

$$Re_t = \frac{u \cdot L}{\nu} = \frac{0.215 \times 1}{10^{-6}} = 2.15 \times 10^5$$

(38%)

Timescale $= \frac{L}{u} = 4.65$ s

(c) If the impeller is switched off, the turbulence decays. Since flow is homogeneous

$$\frac{du^2}{dt} = -\frac{u^3}{L} \quad (1)$$

When $\eta_K = 0.4$ mm, $\varepsilon = \nu^3/\eta_K^4 = 3.9 \times 10^{-5}$ W/kg

Writing $\varepsilon = \frac{u^3}{L}$ gives $u = 0.034$ m/s $(\Rightarrow u^2 = 0.00115$ m^2/s$^2)$

Hence Eq (1) written in terms of $k = u^2$:

$$\frac{dk}{dt} = -\frac{k^{3/2}}{L} \iff \frac{dk}{k^{3/2}} = -\frac{1}{L}\,dt$$

$$\iff \left[-2k^{-\frac{1}{2}}\right]_{k_{init}}^{k_{final}} = -\frac{t}{L}$$

$k_{init} = 0.0464 \ m^2/s^2$ (from part b)

$k_{final} = 0.00115 \ m^2/s^2$

$L = 1 \ m$

$$\Rightarrow \quad \underline{\underline{t = 49.7 \ s}}$$ (40%)

<u>Note:</u> In part (b), the velocity scale u has been taken as (kinetic energy)$^{1/2}$, consistent with Eq (1) (part (c)).

3. (a) Homogeneous in the mean implies

$$\frac{d\bar{c}_A}{dt} = \bar{\dot{w}}_A \quad ; \quad \frac{d\bar{c}_B}{dt} = \bar{\dot{w}}_B \qquad (1,2)$$

$$\frac{d\sigma_A^2}{dt} = -\frac{2\sigma_A^2}{T_{turb}} + 2\overline{\dot{w}c_A'} \qquad (3)$$

$$\frac{d\sigma_B^2}{dt} = -\frac{2\sigma_B^2}{T_{turb}} + 2\overline{\dot{w}c_B'} \qquad (4)$$

$$\dot{w}_A = -k c_A^2 \Rightarrow \bar{\dot{w}} = -k\bar{c}_A^2 - k\sigma_A^2 \qquad (5)$$

Neglecting fluctuations, Eqs (1) & (2) become:

$$\frac{d\bar{c}_A}{dt} = -k\bar{c}_A^2 \Leftrightarrow \frac{d\bar{c}_A}{\bar{c}_A^2} = -k\,dt$$

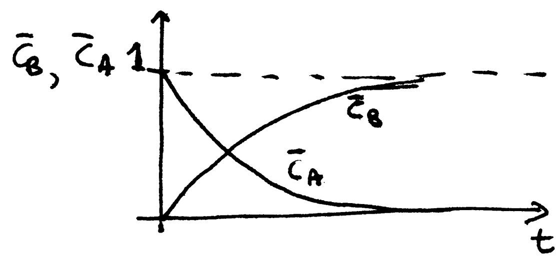
$$\Leftrightarrow \left[-\frac{1}{\bar{c}_A}\right]_1^{\bar{c}_A} = -kt$$

$$\Leftrightarrow \boxed{\bar{c}_A = \frac{1}{1+kt}}$$

$$\frac{d\bar{c}_B}{dt} = +k\bar{c}_A^2 \Leftrightarrow d\bar{c}_B = \frac{k\,dt}{(1+kt)^2}$$

$$\Leftrightarrow \bar{c}_B = \left[-\frac{1}{1+kt}\right]_0^t$$

$$\Leftrightarrow \boxed{\bar{c}_B = 1 - \frac{1}{1+kt}}$$



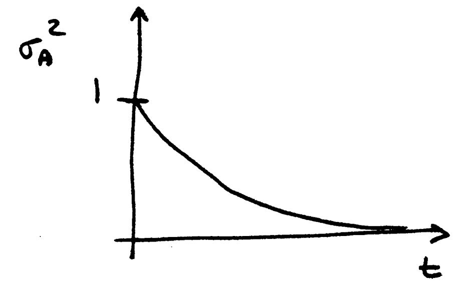
(60%)

(b) The fluctuations of A can be given by Eq 3.

Neglecting chemistry,

$$\frac{d\sigma_A^2}{dt} = -\frac{2\sigma_A^2}{T_{turb}} \iff \sigma_A^2 = \sigma_A^2(t=0)\exp\left(-\frac{2t}{T_{turb}}\right)$$

$$\Rightarrow \sigma_A^2 = \exp\left(-\frac{2t}{T_{turb}}\right)$$



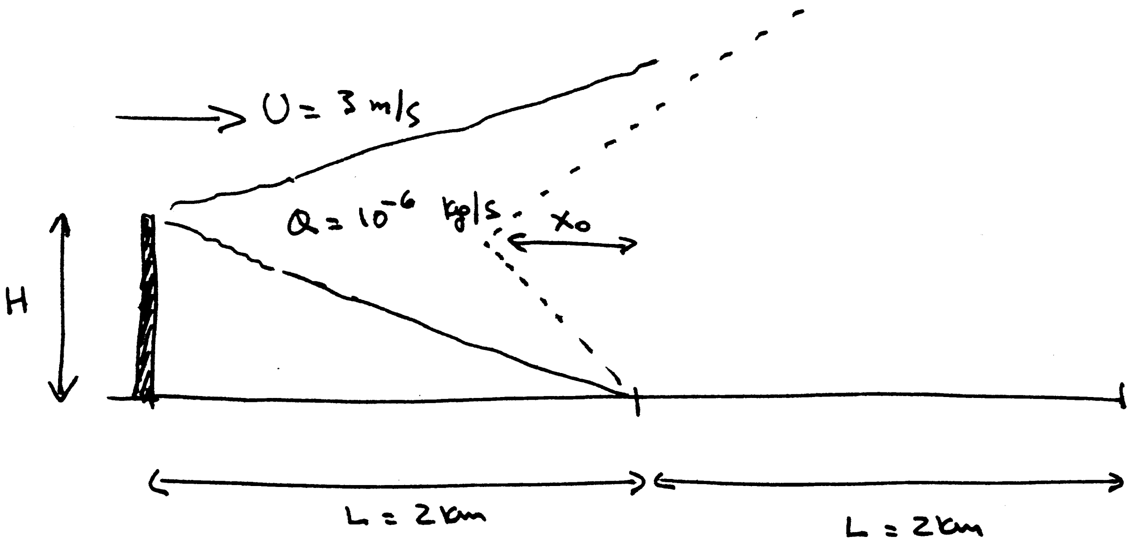
(20%)

(c) If fluctuations are included,

$$\frac{d\bar{c}_A}{dt} = -k\bar{c}_A^2 - k\sigma_A^2$$

Therefore $\bar{c}_A$ will decay <u>faster</u> in the presence of fluctuations. If $kT_{turb} \ll 1$, then the fluctuations will have decayed before chemistry is significant and ignoring fluctuations is a good assumption. (20%)

4.



$\sigma = 0.02x$ $\qquad$ $\sigma = 0.1x$

$(\sigma_y = \sigma_z = \sigma)$

Ground concentration $(z=0)$ at centreline of plume $(y=0)$ is (with ground effect, from Data Card)

$$\Phi(0,0,0) = \frac{Q}{2\pi U \sigma^2} \cdot 2\exp\left(-\frac{H^2}{2\sigma^2}\right) \qquad (1)$$

At end of rural section, $\sigma = 0.02 \times 2000 = 40$ m

The plume in the urban section therefore will behave as if it had originated at $x_0 = \frac{40}{0.1} = 400$m upstream of the beginning of the urban section

So, to calculate $\Phi(x,0,0)$ we use Eq (1)

with $\sigma = 0.1x$ & $x = 400 + 2000$ m $= 2400$ m.

$$\Rightarrow \Phi = \frac{10^{-6}}{\pi \cdot 3 \cdot (0.1)^2 (2400)^2} \exp\left(-\frac{100^2}{2\cdot(0.1\times 2400)^2}\right) \frac{kg}{m^3}$$

$= 1.7 \times 10^{-12}$ kg/m^3 $\qquad$ (70%)

(b) Dispersion above a city is more vigorous because the turbulence intensity $\frac{u'}{U}$ is higher. This is because of extra turbulence generation due to the rough surface of the city. A city is also more likely to have unstable conditions due to heat sources (e.g. cars, domestic heating etc.)

(30%)