ENGINEERING TRIPOS PART IIB

Wednesday 4^{th} May 2005 2.30 to 4

Module 4A10

FLOW INSTABILITY

Solutions

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

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1

(a) If $f_1(\infty) \rightarrow 0$ then $B_1 = 0$ and if $f_2(-\infty) \rightarrow 0$ then $A_2 = 0$. Otherwise f_1 and f_2 would be infinite at these values of z.

Substituting for ϕ_1 and η into the first boundary condition gives (at $z = 0$):

$$\begin{aligned} \tfrac{\partial \phi_1}{\partial z} &= \tfrac{\partial \eta}{\partial t} + U_1 \tfrac{\partial \eta}{\partial x} \\ \Rightarrow \quad -kA_1 e^{(st+ikx)} &= s\eta_0 e^{(st+ikx)} + U_1(ik)\eta_0 e^{(st+ikx))} \\ \Rightarrow \quad -kA_1 &= s\eta_0 + ikU_1\eta_0 \end{aligned}$$

and substituting for ϕ_2 and η into the second boundary condition gives:

$$\begin{aligned} \tfrac{\partial \phi_2}{\partial z} &= \tfrac{\partial \eta}{\partial t} + U_2 \tfrac{\partial \eta}{\partial x} \\ \Rightarrow \quad kB_2 e^{(st+ikx)} &= s\eta_0 e^{(st+ikx)} + U_2(ik)\eta_0 e^{(st+ikx))} \\ \Rightarrow \quad kB_2 &= s\eta_0 + ikU_2\eta_0 \end{aligned}$$

Eliminating η_0 between these two boundary conditions gives:

$$\frac{A_1}{B_2} = -\frac{(s + ikU_1)}{(s + ikU_2)}$$

[20%]

(b) The pressure has to be calculated in each fluid independently because of the vorticity concentrated at the interface. The unsteady Bernoulli equation can be used (data sheet):

$$\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + gz + \frac{\partial \phi}{\partial t} = \text{const.}$$

Furthermore, we eliminate small terms so that the inertial term at $z = 0$ takes the form:

$$|\mathbf{u}|^2 = U^2 + 2Uikf(0)e^{(st+ikx)}$$

Hence:

$$\begin{aligned} p_1 &= (p_{\infty,1} + \frac{1}{2}\rho_1 U_1^2) - (\rho_1 U_1 ikA_1 + \rho_1 g\eta_0 + \rho_1 sA_1)e^{(st+ikx)} \\ p_2 &= (p_{\infty,2} + \frac{1}{2}\rho_2 U_2^2) - (\rho_2 U_2 ikB_2 + \rho_2 g\eta_0 + \rho_2 sB_2)e^{(st+ikx)} \end{aligned}$$

At the interface, $p_1 = p_2$. When there are no perturbations, this means that

$$(p_{\infty,1} + \frac{1}{2}\rho_1 U_1^2) = (p_{\infty,2} + \frac{1}{2}\rho_2 U_2^2)$$

Hence, when there are perturbations, $p_1 = p_2$ requires that:

$$(\rho_1 U_1 ikA_1 + \rho_1 g\eta_0 + \rho_1 sA_1) = (\rho_2 U_2 ikB_2 + \rho_2 g\eta_0 + \rho_2 sB_2)$$

but, from part (a), $\eta_0(s + ikU_2) = B_2 k$, so:

$$\rho_1 A_1(U_1 ik + s) = \rho_2 B_2(U_2 ik + s) + \frac{g(\rho_2 - \rho_1)B_2 k}{(s + ikU_2)}$$

Hence the other expression for A_1/B_2 which must be satisfied at the interface is:

$$\frac{A_1}{B_2} = \frac{\rho_2(U_2 ik + s)}{\rho_1(U_1 ik + s)} + \frac{g(\rho_2 - \rho_1)k}{\rho_1(s + ikU_2)(s + ikU_1)}$$

[50%]

(c) Combining these boundary conditions produces the following expression for s/k:

$$\frac{s}{k} = -i\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \sqrt{\frac{\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{g(\rho_2 - \rho_1)}{k(\rho_2 + \rho_1)}}$$

This is unstable when the term inside the square root is positive:

$$\frac{\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} > \frac{g(\rho_2 - \rho_1)}{k(\rho_2 + \rho_1)}$$

which can be expressed in terms of wavenumber:

$$k > \frac{g(\rho_2^2 - \rho_1^2)}{(U_1 - U_2)^2\rho_1\rho_2}$$

The flow is always unstable to high k, i.e. to short wavelengths. This is a defect of assuming an infinitely-thin shear layer. In reality there will be a cut-off wavelength set by the shear layer thickness. [10%]

(d) The flow is unstable at the lip of the loch so waves will be generated in that region. Away from the lip, waves are neutrally stable. In terms of the above equation, s has zero real component. This solution corresponds to travelling waves, very similar to waves on the surface of the sea. Note that this does *not* mean that waves will be damped in time. Thus waves will be generated at the lip and will travel out to sea along the interface[1]. The oil platform will move up and down accordingly. [20%]

[1]These are called gravity waves. A full analysis can be found in Drazin and Reid 2nd edition §44 (shelf-mark TA.471)

2

(a) Ring A (inner ring): angular momentum = $r_1 v_1 \delta m$; kinetic energy = $(v_1^2 \delta m)/2$.
Ring B (outer ring): angular momentum = $r_2 v_2 \delta m$; kinetic energy = $(v_2^2 \delta m)/2$. [10%]

(b) Final velocity of ring A, having moved to the outside = $r_1 v_1 / r_2$.
Final velocity of ring B, having moved to the inside = $r_2 v_2 / r_1$.
Therefore the final kinetic energy is:

$$\frac{1}{2}\left\{\left(\frac{r_1 v_1}{r_2}\right)^2 + \left(\frac{r_2 v_2}{r_1}\right)^2\right\}\delta m$$

[10%]

(c) Consider the change in kinetic energy, ΔK.E., when the fluids swap places. If this is negative, energy has been released from the mean flow and perturbations may be unstable. On the other hand, if this is positive, energy is required to swap the rings over and perturbations will be stable.

$$\begin{aligned}
\Delta\text{K.E.} &= \frac{1}{2}\left\{\left(\frac{r_1 v_1}{r_2}\right)^2 + \left(\frac{r_2 v_2}{r_1}\right)^2 - v_1^2 - v_2^2\right\}\delta m \\
&= \frac{1}{2}\left\{\frac{r_1^2(r_1^2 v_1^2) + r_2^2(r_2^2 v_2^2) - r_1^2 r_2^2 v_1^2 - r_1^2 r_2^2 v_2^2}{r_1^2 r_2^2}\right\}\delta m \\
&= \frac{1}{2}\left\{\frac{r_2^2 v_2^2(r_2^2 - r_1^2) - r_1^2 v_1^2(r_2^2 - r_1^2)}{r_1^2 r_2^2}\right\}\delta m \\
&= \frac{1}{2}\left\{\frac{r_2^2 - r_1^2}{r_1^2 r_2^2}\right\}\frac{(\Gamma_2^2 - \Gamma_1^2)}{4\pi^2}\delta m
\end{aligned}$$

The term in braces {} is always positive because $r_2 > r_1$. Hence ΔK.E. will always be positive, and the flow will therefore always be stable, if $\Gamma_2^2 > \Gamma_1^2$ [20%]

(d) Kinetic energy of ring A (inner ring) = $(v_1^2 + u_1^2)\delta m/2$.
Kinetic energy of ring B (outer ring) = $(v_2^2 + u_2^2)\delta m/2$. [10%]

(e) Final azimuthal velocity of ring A = $r_1 v_1 / r_2$. Final azimuthal velocity of ring B = $r_2 v_2 / r_1$. Final axial velocity of both rings = $(u_1 + u_2)/2$ by conservation of axial momentum.

Again, consider the change in kinetic energy, ΔK.E., when the fluids swap places and exchange some axial momentum. Note that, in coming to the same axial velocity, the fluids have released the maximum possible amount of kinetic energy from their axial motion. This will be the most unstable situation.

$$\begin{aligned}\Delta\text{K.E.} &= \frac{1}{2}\left\{\frac{r_2^2 - r_1^2}{r_1^2 r_2^2}\frac{(\Gamma_2^2 - \Gamma_1^2)}{4\pi^2} + \frac{(u_1+u_2)^2}{2} - u_1^2 - u_2^2\right\}\delta m \\ &= \frac{1}{2}\left\{\frac{r_2^2 - r_1^2}{r_1^2 r_2^2}\frac{(\Gamma_2^2 - \Gamma_1^2)}{4\pi^2} - \frac{u_2^2 - 2u_1u_2 + u_1^2}{2}\right\}\delta m \\ &= \frac{1}{2}\left\{\frac{r_2^2 - r_1^2}{r_1^2 r_2^2}\frac{(\Gamma_2^2 - \Gamma_1^2)}{4\pi^2} - \frac{(u_2-u_1)^2}{2}\right\}\delta m \qquad (1)\end{aligned}$$

Hence the flow will be stable[2] when:

$$\frac{r_2^2 - r_1^2}{r_1^2 r_2^2}\frac{(\Gamma_2^2 - \Gamma_1^2)}{4\pi^2} \geq \frac{(u_2-u_1)^2}{2}$$

The axial velocity difference is always destabilising. [40%]

(f) The blood in main arteries is at the point of transition to turbulence. We are searching for a mechanism to delay transition. The initial stage of transition to turbulence is the amplification of small perturbations. We have discovered in part (e) that an axial velocity difference (i.e. shear) has a destabilising effect but that rotation has a stabilising effect if $\Gamma_2^2 > \Gamma_1^2$. There will always be destabilising shear in an artery. At low Reynolds number, the flow is stabilised by viscosity. Around transition, however, the destabilising effect of shear takes over. Another way to stabilise the flow would be to spin the blood near the walls of the artery. This would delay the transition to turbulence. It could be achieved by etching helical grooves (rifling) into the walls of the material. Of course, where the rifling suddenly stops, at the end of the graft, the flow will be less stable and this may cause problems[3]. [10%]

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[2]A formal derivation of this result is found in Howard & Gupta (1962) *J. Fluid Mech.* **14** 463 - 476 §2
[3]For a full review of blood flow in arteries see Ku (1997) *Ann. Rev. Fluid Mech.* **29** 399-434

3

(a) The term proportional to U^2 is the centrifugal force of the fluid on the pipe. As the fluid goes round a corner in the pipe ($d^2Y/dx^2 \neq 0$) the fluid accelerates. The reactive force acts at the point of curvature and makes sinuous perturbations more unstable. [10%]

(b) The term proportional to U is the Coriolis force. As the angle of the pipe changes in time ($d^2Y/dtdx \neq 0$) the fluid accelerates. In sinuous oscillations of a pipe, the Coriolis force is stabilising but is never as stabilising as the centrifugal force is destabilising. [10%]

A further 10% is given for more discussion about these forces, especially if this shows evidence of reading around the subject. Although separated conceptually, these forces both have the same origin: the acceleration of the fluid within the pipe. This can be readily shown be expanding out the material derivative of the acceleration of a fluid particle: D^2Y/Dt^2. In addition, it is simplistic to say that the Coriolis force is stabilising. It is stabilising for sinuous travelling waves, but in cantilevered pipes it is destabilising. [10%]

(c) The equation of motion without the Coriolis term reduces to:

$$EI\frac{\partial^4 Y}{\partial x^4} + (pA + \rho AU^2 - T)\frac{\partial^2 Y}{\partial x^2} + (\rho A + m)\frac{\partial^2 Y}{\partial t^2} = 0$$

Substituting $Y = Y_0 e^{st} \sin kx$ reduces the equation to:

$$EIk^4 - k^2(pA + \rho AU^2 - T) + s^2(\rho A + m) = 0$$

Solving for s gives:

$$s = \pm\sqrt{\frac{(pA + \rho AU^2 - T)k^2 - EIk^4}{(\rho A + m)}}$$

The pipe is unstable when s has a positive real solution, i.e. when the term in the square root is positive. This is when:

$$U^2 > \frac{EIk^2 - pA + T}{\rho A}$$

It now remains to express k in terms of the length of the pipe. The smallest value of k, given the boundary conditions is π/L, corresponding to the mode shape shown in the diagram. We ignore the small change in L due to the movement of the pipe (which would be slightly stabilising). Hence the critical velocity is:

$$U_{crit}^2 > \frac{EI(\pi/L)^2 - (pA - T)}{\rho A}$$

and since the pipe is discharging without a nozzle, $(pA - T) = 0$. [45%]

(d) A mass balance gives $\rho AU = \rho A_e U_e$, where subscript e denotes exit conditions. The nozzle halves the flow area so the velocity doubles. Two things happen: the internal pressure increases and the tension in the pipe increases. These two factors do not cancel out. A momentum balance (remembering that the atmospheric pressure acts over area A, not just A_e) gives:

$$\begin{aligned} & p_{pipe}A + \rho AU^2 = p_{atm.}A + \rho AUU_e + T \\ \Rightarrow \quad & (p_{pipe} - p_{atm.})A - T = \rho AU(U_e - U) \\ \Rightarrow \quad & pA - T = \rho AU^2 \end{aligned}$$

Therefore the velocity at which the pipe will buckle divides by $\sqrt{2}$:

$$U_{crit}^2 > \frac{1}{2}\frac{EI(\pi/L)^2}{\rho A}$$

[25%]

4

(a) One side of the boat catches the wind slightly, giving it a sideways force. The boat starts to move in this sideways direction. To start with, the mooring rope gives no sideways force. The aerodynamic centre is upstream of the centre of mass, causing the boat to rotate such that more of that side of the boat catches the wind. The aerodynamic force increases. This continues until the boat has moved some distance to the side. Eventually, the force from the mooring rope starts to pull the front of the boat back towards the centreline. The centre of mass continues to move in the original direction, so the boat starts rotates in the opposite direction. Eventually, it presents the other side to the wind and the process repeats itself. This motion can be seen on boats in harbours, with a period of around a minute. [20%]

(b) Substituting for y and θ into the equations of motion gives:

$$\begin{aligned} ms^2Y_0 + S_x s^2\theta_0 + kY_0 &= -q\theta_0 \\ Is^2\theta_0 + s_x Y_0 s^2 &= -qc_a\theta_0 \end{aligned}$$

which can be expressed as:

$$\begin{pmatrix} ms^2+k & S_x s^2 + q \\ S_x s^2 & Is^2 + qc_a \end{pmatrix} \begin{pmatrix} Y_0 \\ \theta_0 \end{pmatrix} = 0$$

[20%]

(c) This will have non-trivial solutions when the determinant is zero:

$$(ms^2+k)(Is^2+qc_a) - S_x s^2(S_x s^2 + q) = 0$$
$$\Rightarrow \quad (mI - S_x^2)s^4 + (kI + qmc_a - S_x q)s^2 + qc_a k = 0$$

Hence

$$\begin{aligned} C_0 &= (mI - S_x^2) \\ C_2 &= (kI + qmc_a - S_x q) \\ C_4 &= qc_a k \end{aligned}$$

[10%]

(d)

$$\begin{aligned} C_0 &= (mI - S_x^2) = \frac{1}{8}S_x^2 \\ C_2 &= kI - mq\left(\frac{S_x}{m} - C_a\right) = kI - mqL \\ C_4 &= qc_a k \end{aligned}$$

where L (positive) is the distance between the aerodynamic centre and the centre of mass. C_0, and C_4 are positive. C_2 can be positive or negative. Solving for s gives:

$$s = \pm\left(-\frac{C_2}{2C_0} \pm \sqrt{\frac{C_2^2 - 4C_0C_4}{4C_0^2}}\right)^{1/2}$$

If C_2 is positive *and* $C_2^2 > 4C_0C_4$ then s is pure imaginary and the situation is stable. (The system supports oscillations but their amplitude does not increase in time and, were we to include damping, they would die away). If C_2 is positive then s only has a real component when $C_2^2 < 4C_0C_4$, which is one condition for instability. If C_2 is negative, we have a second condition for instability. [20%]

(e) When $(kI - mqL)$ is positive, the system is unstable if:

$$\begin{aligned} & 4C_0C_4 > C_2^2 \\ \Rightarrow \quad & \tfrac{1}{2}S_x^2 qc_a k > (kI - mqL)^2 \\ \Rightarrow \quad & \tfrac{1}{2}\rho c \left.\tfrac{\partial C_L}{\partial \alpha}\right|_0 U^2 > \tfrac{2(kI - mqL)^2}{c_a k} \end{aligned}$$

Increasing L, the distance between the aerodynamic centre and the centre of mass, makes the situation less stable. The system is also unstable when $kI < qmL$, which reduces to:

$$\frac{1}{2}\rho c \left.\frac{\partial C_L}{\partial \alpha}\right|_0 U^2 > \frac{kI}{mL}$$

Again, increasing L makes the situation less stable. [20%]

(f) In a more detailed analysis, one should take into account:

- the added mass and added moment of inertia of the submerged portion of the boat;
- wave radiation, which will act as added damping.

[10%]

END OF PAPER

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