MODULE 4A12 ; TURBULENCE
FRIDAY 6 MAY, 2005 9 - 10.30

(1) $$\frac{Dk}{Dt} = \frac{\partial k}{\partial t} + U\frac{\partial k}{\partial x} + \ldots\ldots$$

Total or material derivative ; change due to temporal and spatial variation ; change following a fluid element

$\nu_t \left[\frac{\partial U_i}{\partial x_j}\right]^2$ Production of turbulent kinetic energy from product of shear stress and velocity gradient

$\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)$ Turbulent diffusion of kinetic energy

ε Dissipation of turbulent kinetic energy to internal energy

$$-\overline{u'v'} = u_*^2 \qquad \frac{\partial U}{\partial y} = u_*/\chi y \qquad \varepsilon = u_*^3/\chi y$$

$$\nu_T = C_\mu k^2/\varepsilon = \cancel{\chi} u_* \cancel{y} = C_\mu k^2 \cancel{\chi y}/u_*^3$$

$$\therefore C_\mu = u_*^4/k^2 = (-0.3)^2 = 0.09$$

50%

$\underbrace{k \sim x^{-1}}$ turbulent kinetic energy

$L \sim (x+c)^{1/2}$

$E(k_w) = A\,\varepsilon^{2/3}\,\underbrace{k_w^{-5/3}}_{\text{wavenumber}}$

$$\varepsilon \sim k^{3/2}/L \sim \frac{x^{-3/2}}{x^{1/2}} \sim x^{-2}$$

$$E(k_w) = A\, x^{-4/3}\, k_w^{-5/3}$$

$$u'^2 \sim x^{-1}$$

→ decays faster ;
Relatively more energy in the larger eddies

50%

(2) Kinematic momentum flux / unit width

$M_o' = U^2 b$ = constant

$\frac{d}{dx} Ub = 2u_e = 2\alpha U$ 10%, 10%

$b \sim x^1 \; ; \; l \sim b$ 20%

$U \sim x^{-1/2}$

$\varepsilon \sim u'^3/b \sim U^3/b \sim \frac{x^{-3/2}}{x} \sim x^{-5/2}$ 20%

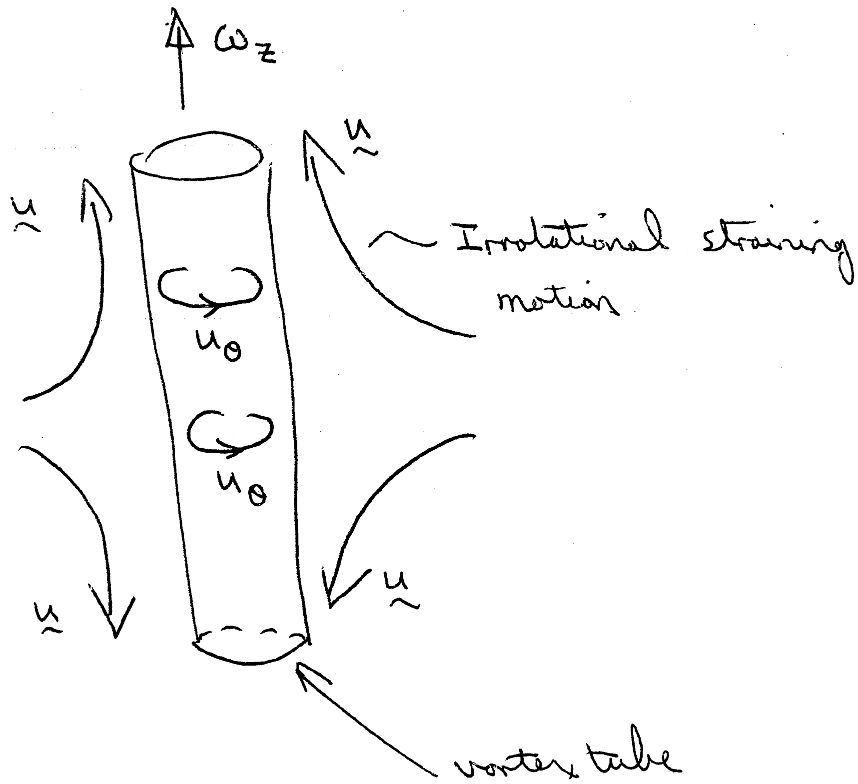
$\eta \sim (\nu^3/\varepsilon)^{1/4} \sim (x^{5/2})^{1/4} \sim x^{5/8}$ 20%

Jet properties independent of position?
Obtained by integrating ~~across~~ variables over cross section i.e. by definition of similarity arguments.

u'/U = constant.

Required on dimensional grounds and by local similarity. 20%

1.3 (a)



Burger's vortex is thought to be a good model of the smallest scales in turbulence, called "worms".

(b)
$$(\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u}_\omega = \omega_z \frac{\partial}{\partial z}(\underset{\sim}{u}_\omega) = 0$$

($\underset{\sim}{u}_\omega$ <u>not</u> a function of z)

$$(\underset{\sim}{u}_\omega \cdot \nabla)\underset{\sim}{\omega} = u_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(\underset{\sim}{\omega}) = 0$$

($\underset{\sim}{\omega}$ <u>not</u> a function of θ)

3 (c) $(\underset{\sim}{u}\cdot\nabla)\,\underset{\sim}{\omega} = u_x \dfrac{\partial \omega_z}{\partial x}\,\hat{\underset{\sim}{e}}_x = (-\alpha x)\left(-\dfrac{2x}{\delta^2}\right)\omega_z \hat{\underset{\sim}{e}}_z$

$$(\underset{\sim}{\omega}\cdot\nabla)\,\underset{\sim}{u} = \omega_z \frac{\partial \underset{\sim}{u}}{\partial z} = \omega_z \alpha \hat{\underset{\sim}{e}}_z$$

$$\nu \nabla^2 \underset{\sim}{\omega} = \nu \frac{\partial^2 \omega_z}{\partial x^2}\hat{e}_z = \nu\left[\left(\frac{-2x}{\delta^2}\right)^2 \omega_z - \frac{2}{\delta^2}\omega_z\right]\hat{\underset{\sim}{e}}_z$$

But $(\underset{\sim}{\omega}\cdot\nabla)\,\underset{\sim}{u} + \nu\nabla^2\underset{\sim}{\omega} = (\underset{\sim}{u}\cdot\nabla)\,\underset{\sim}{\omega}$

$$\Rightarrow \quad \alpha\omega_z + \nu\left[\left(\frac{-2x}{\delta^2}\right)^2 - \frac{2}{\delta^2}\right]\omega_z = \frac{2\alpha x^2}{\delta^2}\omega_z$$

$$\Rightarrow \quad \alpha + \nu\left[\frac{4x^2}{\delta^4} - \frac{2}{\delta^2}\right] = \frac{2\alpha x^2}{\delta^2}$$

$$\Rightarrow \quad \alpha - \frac{2\nu}{\delta^2} = \frac{2x^2}{\delta^2}\left[\alpha - \frac{2\nu}{\delta^2}\right]$$

This is satisfied if

$$\alpha = \frac{2\nu}{\delta^2}$$

$$\Rightarrow \quad \underline{\underline{\delta = \sqrt{2\nu/\alpha}}}$$

(d) If $u_s = 0$ then

$$\frac{\partial \omega}{\partial t} = \nu\nabla^2\omega$$

The vortex sheet will then thicken by diffusion, at a rate $\delta \sim (\nu t)^{1/2}$.

4 (a) (i) The vortex lines are 'glued' into the fluid.

(ii) The flux of vorticity down a vortex tube is the same at all cross-sections and independent of time.

4. (b)

$$\frac{D}{Dt}(\underline{u}\cdot\underline{\omega}) = \underline{u}\cdot\frac{D\underline{\omega}}{Dt} + \underline{\omega}\cdot\frac{D\underline{u}}{Dt}$$

$$= \underline{u}\cdot\left[(\underline{\omega}\cdot\nabla)\underline{u}\right] + \underline{\omega}\cdot\left[-\nabla(p/\rho)\right]$$

$$= (\underline{\omega}\cdot\nabla)(u^2/2) - \nabla\cdot(\underline{\omega}\, p/\rho)$$

$$= \nabla\cdot(\underline{\omega}\, u^2/2) - \nabla\cdot(\underline{\omega}\, p/\rho)$$

$$= \nabla\cdot\left[(u^2/2 - p/\rho)\,\underline{\omega}\right]$$

$$\frac{D}{Dt}(\underline{u}\cdot\underline{\omega})\,\delta V = \delta V\,\frac{D}{Dt}(\underline{u}\cdot\underline{\omega}) \qquad \text{since } \frac{D}{Dt}\delta V = 0$$

$$\Rightarrow \quad \frac{D}{Dt}\int(\underline{u}\cdot\underline{\omega})\,dV = \int\frac{D}{Dt}(\underline{u}\cdot\underline{\omega})\,dV$$

$$= \int\nabla\cdot\left[\left(\frac{u^2}{2}-\frac{p}{\rho}\right)\underline{\omega}\right]dV$$

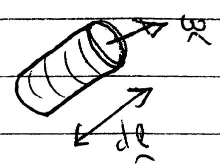
$$= \oint\left(\frac{u^2}{2}-\frac{p}{\rho}\right)\underline{\omega}\cdot d\underline{s}$$

$$= 0$$

(c)

$$\underline{\omega}\,dV = \underline{\omega}\,A\,dl = |\underline{\omega}|A\,d\underline{l} = \Phi\,d\underline{l}$$

Thus, for one tube



$$H = \int \underline{u}\cdot\underline{\omega}\,dV = \oint_C \underline{u}\cdot(\Phi\,d\underline{l})$$

For both tubes,
$$H = \oint_{C_1}\underline{u}\cdot(\Phi_1\,d\underline{l}) + \oint_{C_2}\underline{u}\cdot(\Phi_2\,d\underline{l})$$

Since Φ_1 and Φ_2 are constant around the tubes, Stokes' theorem gives us,

$$H = \Phi_1\oint_{C_1}\underline{u}\cdot d\underline{l} + \Phi_2\oint_{C_2}\underline{u}\cdot d\underline{l} = \Phi_1\Phi_2 + \Phi_2\Phi_1$$

4 (d) If $\underset{\wedge}{\omega}$ in one tube is reversed the linkage of the tubes becomes left-handed, so the application of Stokes theorem introduces a minus sign and $H = -2\Phi_1\Phi_2$

4 (e) H is not conserved if the fluid is viscous or subject to a buoyancy force because the vortex lines are no longer frozen into the fluid so that the linkage of the two tubes is not preserved.