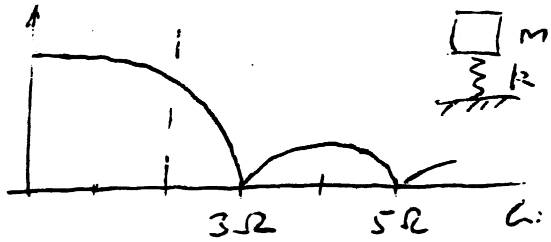
4C6 2005

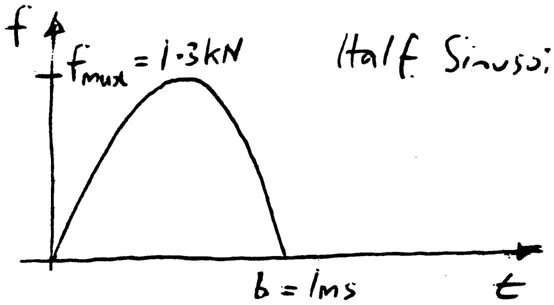
1 (a) (i) In order to excite all frequencies up to 1 kHz it is sensible to have the first zero of the impulse spectrum at, say, 1.5 kHz



$\therefore\ 3\Omega = 1500 \times 2\pi$ rad/s $\qquad \therefore\ \Omega = 1000\pi$ rad/s $= \sqrt{\frac{k}{m}}$

$\therefore\ k = m\Omega^2 = 0.2 \times (1000\pi)^2$
$= \underline{2\ \text{MN/m}}$ = hammer tip stiffness

(ii)



Half sinusoidal pulse $\quad \therefore\ b = \frac{\pi}{\Omega} = \underline{1\,ms}$

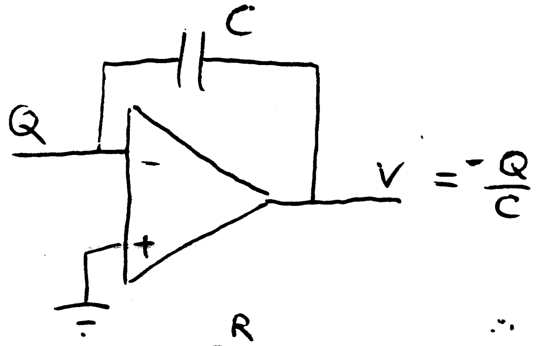
$x = X \sin \Omega t$
$v = X\Omega \cos \Omega t$

$v_{max} = X\Omega = 2\ ms^{-1}$

$\therefore\ X = \frac{2}{1000\pi}$ m

and $f_{max} = kX = 2 \times 10^6\ \frac{2}{1000\pi} = \underline{1.3\ kN}$

(iii)

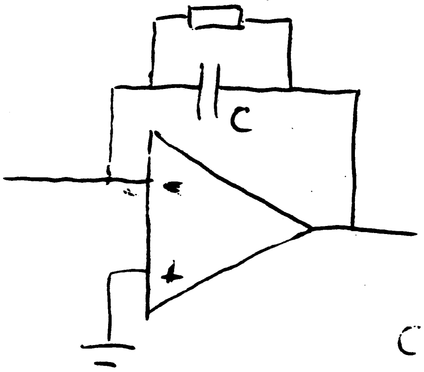


$V = -\frac{Q}{C}$

Peak force = 1.3 kN
$\therefore$ Peak charge $Q = 4 \times 1300$ pC
$= 5200$ pC

Say Peak V = 4 Volts to avoid saturation

$\therefore\ C = \frac{Q}{V} = \frac{5200}{4} = \underline{1300\ pF}$



High pass filter necessary to prevent buildup of dc charge on capacitor due to thermal drift and imbalance in op amp

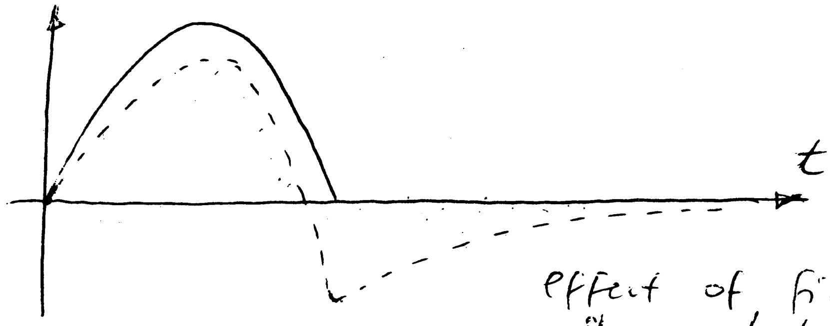
Choose $f_{cut} = 5\,Hz$, say
$= \frac{1}{2\pi}\frac{1}{RC}$

$\therefore\ R = \frac{1}{2\pi C f_{cut}} = \frac{1}{2\pi\ 1300 \times 10^{-12} \times 5}$

$= \underline{24\ M\Omega}$

(iii) cont These component values may not be sensible for various reasons but no discussion of this issue is required.

~~(iv)~~



effect of filter (exaggerated) shown dashed.

There can be no net dc content in the output from the filtered charge amp ∴ the two shaded areas are equal

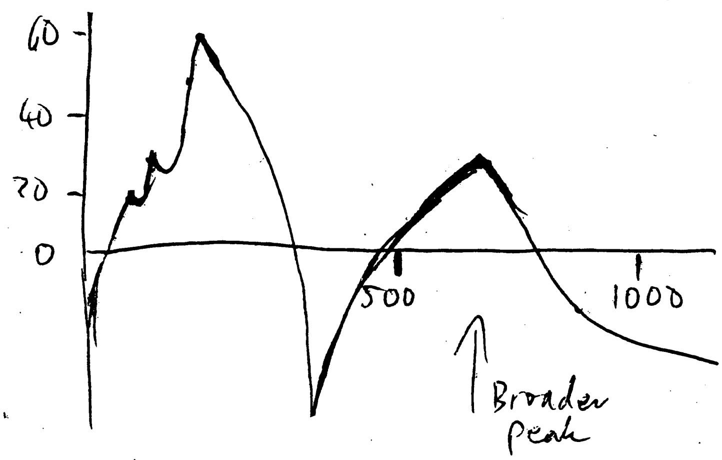
(iv) Sample rate of 4 kHz (ie greater than twice 1·5 kHz) satisfies Nyquist criterion.

(b) From the table use $\Delta f_n = \frac{f_n}{Q_n}$ half power bandwidth

and peak height $= Q_n\, U_n(x)\, U_n(y)$

mode	f_n	Δf_n	peak height (dB)
1	47	1·7	12 (22)
2	56	1·5	−28 (29)
3	152	0·58	728 (57)
4	658	39	28 (29)

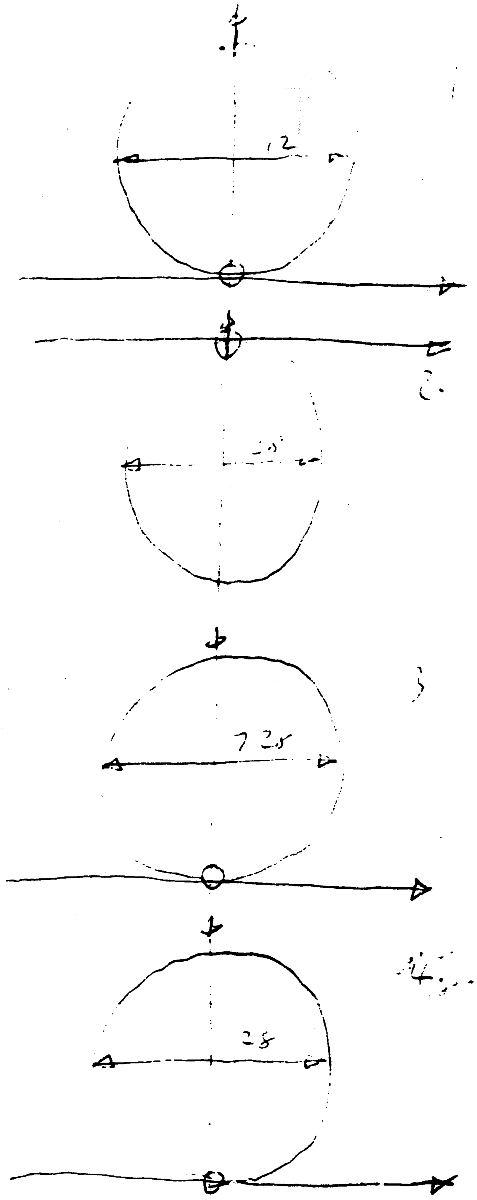
(i)



Pattern of signs is + − + +, so only get one antiresonance.

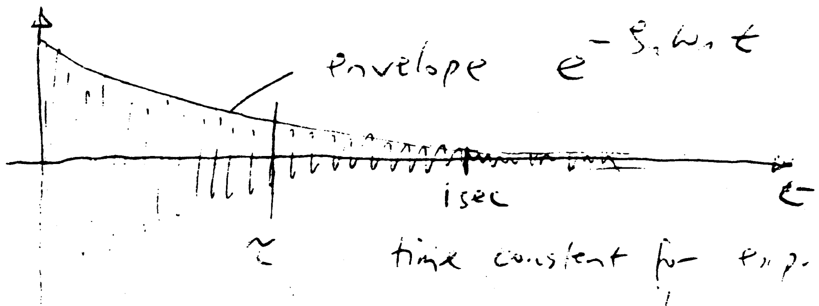
(ii) Modes 1 & 2 are well enough separated because the half power bandwidths are much less than the peak separation

(iii)



The centres of these two circles may be shifted due to proximity of modes

(iv) Mode 3 dominates the response after initial contributions from modes 1, 2 & 4 have died away



$\xi_1 = \frac{1}{2Q_1}$

time constant for exponential decay

$\tau = \frac{1}{\xi_1\omega_1} = 0.54$ seconds

2 (a) Only complex term is $\frac{1}{1+g} \rightarrow K = \frac{1}{1+g(1+i\eta_a)}$

Then $\frac{\mathrm{Im}(EI)}{\mathrm{Re}(EI)} = \frac{-\frac{h_1^2}{8}(E_2h_2 + 2E_3h_3)\,\mathrm{Im}(K)}{E_1h_1^3/12}$

$$= -\frac{3}{2}\,\frac{(E_2h_2 + 2E_3h_3)}{E_1h_1}\,\mathrm{Im}(K)$$

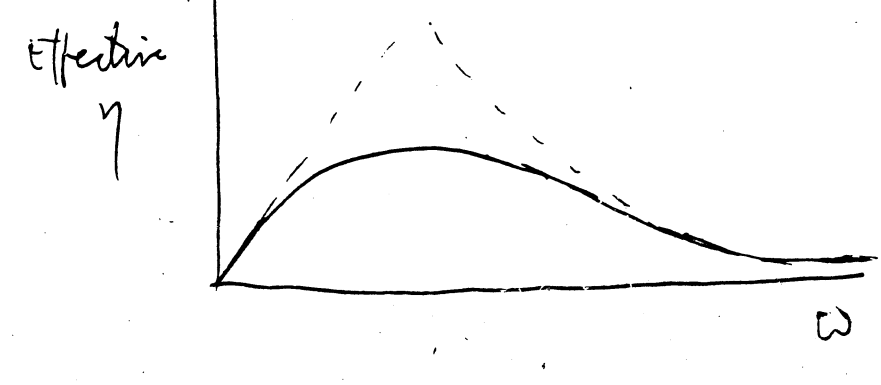
But $K = \frac{1+g(1-i\eta_a)}{(1+g)^2 + g^2\eta_a^2}$

$\therefore \mathrm{Im}\,K = \frac{-g\eta_a}{(1+g)^2 + g^2\eta_a^2}$

(b) Frequency dependence arises from $g = \frac{\lambda}{\omega}$, $\lambda = \frac{G_2}{E_3h_2h_3}\sqrt{\frac{E_1h_1^3}{12m}}$

So $\mathrm{Im}\,K = \frac{-\lambda\omega\eta_a}{(\lambda+\omega)^2 + \lambda^2\eta_a^2}$

As $\omega \to 0$, $\mathrm{Im}\,K \to -\omega\eta_a/\lambda$
As $\omega \to \infty$, $\mathrm{Im}\,K \to -\lambda\eta_a/\omega$
So there must be a minimum of $\mathrm{Im}K$, ie a maximum of damping, at some frequency.



2 (b) cont. Maximum occurs where $\frac{d}{d\omega}(\mathrm{Im}\,K) = 0$

$$\therefore\ [(\lambda+\omega)^2 + \lambda^2\eta_a^2](-\lambda\eta_a) + \lambda\omega\eta_a \cdot 2(\lambda+\omega) = 0$$

$$\therefore\ \omega^2 = \lambda^2(1+\eta_a^2)$$

$$\text{ie}\quad \omega_{max} = \lambda\sqrt{1+\eta_a^2}\ .$$

At this frequency,

$$\mathrm{Im}\,K = \frac{-\lambda^2\eta_a\sqrt{1+\eta_a^2}}{\lambda^2[1+\sqrt{1+\eta_a^2}\,]+\eta_a^2}$$

$$= \frac{-\eta_a\sqrt{1+\eta_a^2}}{1+2\sqrt{1+\eta_a^2}+1+\eta_a^2+\eta_a^2}$$

$$= \frac{-\eta_a}{2(1+\sqrt{1+\eta_a^2})}$$

$\therefore$ Max value of

$$\frac{\mathrm{Im}(EI)}{\mathrm{Re}(EI)} = \frac{3}{4}\frac{(E_2h_2+2E_3h_3)}{E_1h_1}\frac{\eta_a}{1+\sqrt{1+\eta_a^2}}$$

(c) Value of optimum loss factor is independent of λ, i.e. independent of the value of $\mathrm{Re}(G_2)$. However, for the free layer (from notes) the effective loss factor is proportional to $\mathrm{Re}(E_2)$.

So for a free layer we need to choose a stiff material (high E_2) as well as one with high loss factor. These tend to be conflicting requirements, putting a limit on performance.
However, for a constrained layer we are free to choose a "soft" material in order achieve high η_a, so it possible to achieve a much higher performance.

3(a) Try $p = R(r)T(\theta)Z(z)e^{i\omega t}$.
Then the given differential equation is

$$c^2\left[R''TZ + \frac{R'}{r}TZ + \frac{R}{r^2}T''Z + RTZ''\right] = -\omega^2 RTZ$$

Multiply by $\frac{r^2}{RTZ}$ $\rightarrow$ $r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{T''}{T} + r^2\frac{Z''}{Z} = -\frac{\omega^2}{c^2}r^2$

This can be written $\frac{T''}{T}$ = function of r, z only

$\therefore$ $\frac{T''}{T}$ = constant $= -n^2$ say

$\therefore$ $T = \left.\begin{matrix}\sin\\ \cos\end{matrix}\right) n\theta$: only sensible if $n = 0, 1, 2, 3 \ldots$

Now $r^2\frac{R''}{R} + r\frac{R'}{R} - n^2 + r^2\frac{Z''}{Z} = -\frac{\omega^2}{c^2}r^2$

Rearrange : $\underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{n^2}{r^2} + \frac{\omega^2}{c^2}}_{r \text{ only}} = \underbrace{-\frac{Z''}{Z}}_{z \text{ only}}$

$\therefore$ $-\frac{Z''}{Z}$ = constant $= \lambda^2$ say

$\therefore$ $Z = \left.\begin{matrix}\sin\\ \cos\end{matrix}\right) \lambda z$ and $R'' + \frac{1}{r}R' + \left(\frac{\omega^2}{c^2} - \lambda^2 - \frac{n^2}{r^2}\right)R = 0$

If we set $k^2 = \frac{\omega^2}{c^2} - \lambda^2$ this is exactly the same as Bessel's equation derived for the membrane.
So solution is $J_n(kr)$.

So in total, $p = J_n(kr)\left(A\sin\lambda z + B\cos\lambda z\right)\left(\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}\right.$

On the top and bottom, $\frac{\partial p}{\partial z} = 0$ for all r, θ

ie $\lambda(A\cos\lambda z - B\sin\lambda z) = 0$ on $z = 0, h$

$\therefore$ $A = 0$, $\sin\lambda h = 0$ $\rightarrow$ $\lambda h = m\pi$, $m = 0, 1, 2, 3 \ldots$

3 contd.

On the curved surface $\frac{\partial p}{\partial r} = 0$ for $r = a$ and all z, θ.

$\therefore J_n'(ka) = 0$

Frequencies are determined by

$$\lambda^2 = \frac{m^2\pi^2}{h^2} = \frac{\omega^2}{c^2} - k^2 \text{ where } k \text{ satisfies } J_n'(ka) = 0$$

ie if the roots of $J_n'(s) = 0$ are $s_1^{(n)}, s_2^{(n)} \ldots$

then $$\omega^2 = c^2\left[\frac{m^2\pi^2}{h^2} + \left(\frac{s_j^{(n)}}{a}\right)^2\right]$$

From graphs: for $J_0' = 0$, $s_1^{(0)} = 0$, $s_2^{(0)} \simeq 3.8$

for $J_1' = 0$, $s_1^{(1)} \simeq 1.8$, $s_2^{(1)} \simeq 5.3$

for $J_2' = 0$, $s_1^{(2)} \simeq 3.1$

For $m = 0$ and using $s_1^{(0)} = 0$, get $\omega = 0$.

So first non-zero frequency is either

$m = 0$ $s_1^{(1)} \simeq 1.8$ or $m = 1$, $s_1^{(0)} = 0$

so $\omega^2 \simeq c^2\left(\frac{\pi^2}{h^2} + 0\right)$ or $\omega^2 \simeq c^2\left(0 + \left(\frac{1.8}{a}\right)^2\right)$

With $h = 1$, $a = 0.5$, $\frac{\pi^2}{h^2} \simeq 9.9$, $\left(\frac{1.8}{a}\right)^2 \simeq 13.0$

So the lowest non-zero frequency comes from $m = 0$, $n = 1$

Then $\omega = \frac{c\pi}{h}$, $\therefore f = \frac{c}{2h} = 170$ Hz.

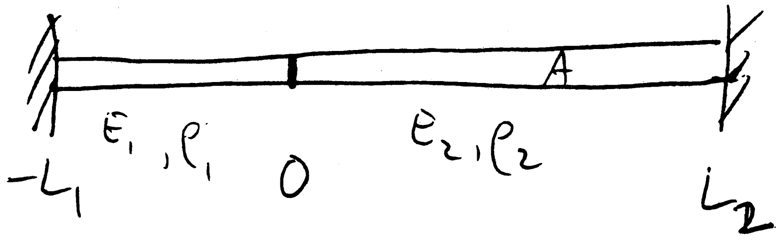
The mode shape is simply $p = \cos\left(\frac{\pi z}{h}\right)$,

independent of r, θ: an "organ pipe mode"

With a small hole, a "Helmholtz air piston" will be able to move in and out in response to internal pressure. The zero frequency found above for $n = 0, m = 0$ is raised to become a Helmholtz resonance.

Fixing the "Helmholtz piston" restores the rigid wall — so adding the hole amounts to removing one constraint. So the new and old frequencies must interlace.

4 (a)



Governing equation for axial vibration

$$E_j \frac{\partial^2 y}{\partial x^2} = \rho_j \frac{\partial^2 y}{\partial t^2} \qquad j = 1, 2$$

For harmonic response $y = u(x)\, e^{i\omega t}$

then $E_j u'' = -\rho_j \omega^2 u$

ie $u'' = -k_j^2 u$, $\quad k_j^2 = \dfrac{\rho_j \omega^2}{E_j}$

Then general solution is $u = \alpha_j \sin(k_j x + \phi_j)$
where α_j, ϕ_j are constants.
But $y = 0$ at $x = -L_1, L_2$
This fixes the required phases ϕ_j, and gives the form as given:

$$u(x) = \begin{cases} \alpha_1 \sin k_1 (L_1 + x) & -L_1 \le x \le 0 \\ \alpha_2 \sin k_2 (\cancel{x - L_2}) & 0 \le x \le L_2 \end{cases}$$

At the join (i) u is continuous $\rightarrow -\alpha_1 \sin k_1 L_1 = \alpha_2 \sin k_2 L_2$ (1)
(ii) force balance across join means

$$A E_1 \left.\frac{du}{dx}\right|_{0-} = A E_2 \left.\frac{du}{dx}\right|_{0+}$$

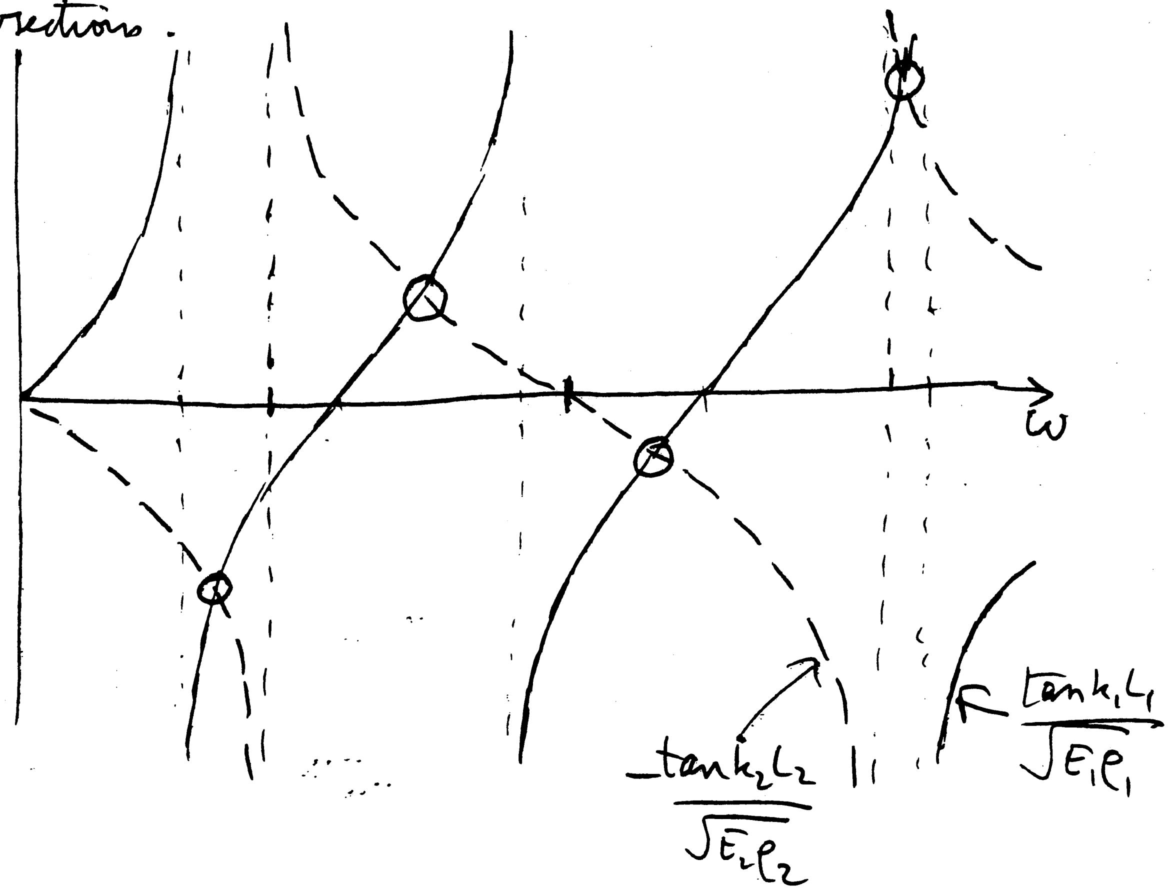
$$\therefore \; E_1 \alpha_1 k_1 \cos k_1 L_1 = E_2 \alpha_2 k_2 \cos k_2 L_2 \quad (2)$$

Divide (1) by (2):
$$\frac{\tan k_1 L_1}{E_1 k_1} = -\frac{\tan k_2 L_2}{E_2 k_2}$$

$$\therefore \; \frac{\tan k_1 L_1}{\sqrt{E_1 \rho_1}} = -\frac{\tan k_2 L_2}{\sqrt{E_2 \rho_2}} \qquad //$$

k_1, k_2 both $\propto \omega$, so this condition determines allowed ω

4 (b) Plot $\frac{\tan k_1 L_1}{\sqrt{E_1 \rho_1}}$, $-\frac{\tan k_2 L_2}{\sqrt{E_2 \rho_2}}$ and look for intersections.



Ringed intersection are the natural frequencies of the coupled system.

(c) If the join is disconnected, there are two separate rods which are both free at the point $x = 0$. The natural frequencies of these two rods are at the asymptotes of the two original tan graphs plotted above. It is clear from the picture that there is one ringed intersection between each pair of asymptotes.

[Same expressions for $y(x,t)$ are OK for the two separated rods, but now the boundary condition is that $\frac{\partial y}{\partial x} = 0$

at $x = 0$ for both rods, so natural frequencies satisfy $\cos k_j L_j = 0$, ie $\tan k_j L_j = \infty$]

4(c) cont.

Alternative way to use interlacing:
Take coupled system and fix the join so there is no motion. This produced two rods which are both fixed at both ends, and no longer coupled. These fixed-fixed rods have natural frequencies at the zeros of the tan graphs: so the ringed intersections interlace the combined zeros, but this time a constraint has been added rather than removed, so the lowest ring is below the lowest zero whereas from the previous argument it is higher than the lowest asymptote.