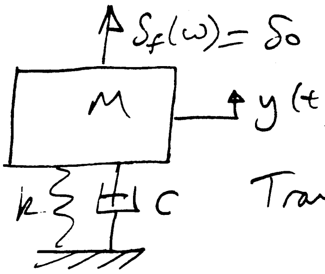
4C7 – 2005

Q1.



$$m\ddot{y} + c\dot{y} + ky = f(t)$$

Transfer function $H(i\omega) = \frac{Y}{F} = \frac{1}{-m\omega^2 + i\omega c + k}$

Output spectral density: $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$

$$\Rightarrow S_y(\omega) = \frac{S_0}{(k - \omega^2 m)^2 + \omega^2 c^2}$$

M. S. Response

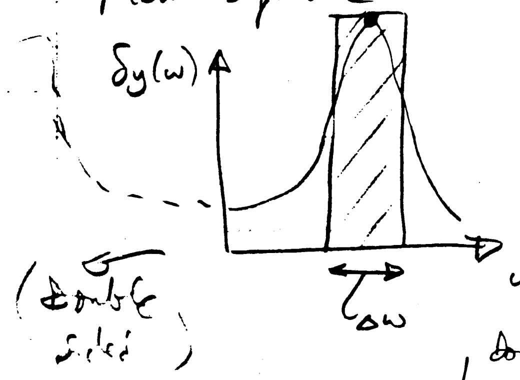
$$E[y^2] = \int_{-\infty}^{\infty} S_y(\omega) = \int_{-\infty}^{\infty} \left| \frac{1}{-m\omega^2 + i\omega c + k} \right|^2 S_0 \, d\omega$$

Using the standard integrals on the data sheet

$B_1 = 0$, $B_0 = \sqrt{S_0}$, $A_2 = m$, $A_1 = c$, $A_0 = k$

$$E[y^2] = \frac{\pi A_0 B_1^2 + A_2 B_0^2}{A_0 A_1 A_2} = \frac{\pi \cancel{m} S_0}{\cancel{m} ck} = \underline{\underline{\frac{\pi S_0}{ck}}}$$

Mean square Bandwidth $\Delta\omega$



Area of rectangle = total area under spectral density

Peak of curve (for light damping set $\omega = \omega_n$

$$\Rightarrow |H(\omega_n)| = \frac{1}{\omega_n c}$$

double-sided

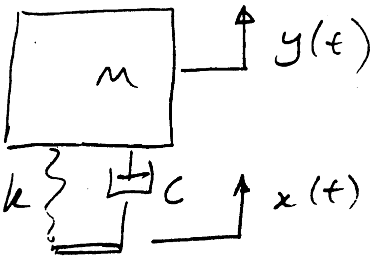
$$E[y^2] = 2 \cdot |H(\omega_n)|^2 S_0 \, \Delta\omega$$

ie $\frac{\pi \cancel{S_0}}{ck} = 2 \left| \frac{1}{\omega_n c} \right|^2 \cancel{S_0} \, \Delta\omega$

$$\Rightarrow \Delta\omega = \frac{\pi}{ck} \frac{c^2}{2} \omega_n^2 = \frac{\pi c}{2m} = \frac{\pi (\zeta \cancel{2} \sqrt{km})}{\cancel{2} m}$$

ie $\underline{\underline{\Delta\omega = \pi \zeta \omega_n}}$

(c)



$$m\ddot{y} + c\dot{y} + ky = c\dot{x} + kx$$

$$H(i\omega) = \frac{Y}{X} = \frac{k + i\omega c}{m(i\omega)^2 + ci\omega + k}$$

$$\sigma_{\dot{y}}^2 = E[\dot{y}^2] = \int_{-\infty}^{\infty} \omega^2 S_{yy}(\omega)\, d\omega$$

At $\omega = \omega_n$, $H(i\omega_n) \approx \frac{k}{i\omega_n c}$ & $S_{xx}(\omega_n) = \frac{S_1}{1 + (\omega_n/\omega_0)^2}$

So $\sigma_{\dot{y}}^2 \approx 2\omega_n^2 |H(i\omega_n)|^2 S_{xx}(\omega_n) \Delta\omega$

$$= 2\frac{k}{m}\left|\frac{k}{i\omega_n c}\right|^2 \frac{S_1}{1 + (\omega_n/\omega_0)^2}\, \pi\zeta\omega_n$$

$$= 2\frac{k}{\cancel{m}}\left(\frac{\cancel{m}k}{c^{\cancel{2}}}\right) \frac{S_1}{1 + \frac{k}{m\omega_0^2}}\, \frac{\pi \cancel{c}}{2m}$$

$$= \frac{2k^2\pi S_1}{cm\left(1 + \frac{k}{m\omega_0^2}\right)} \qquad (1)$$

swap

(b) The probability distribution of peaks for a narrow band process is $p_p(a) = \frac{a}{\sigma_{\dot{y}}^2} e^{-a^2/2\sigma_{\dot{y}}^2}$ (data sheet)

So the probability that a peak exceeds a is

$$\int_a^\infty p_p(a)\, da = \int_a^\infty \frac{a}{\sigma_{\dot{y}}^2} e^{-a^2/2\sigma_{\dot{y}}^2}\, da$$

$$= \left[-e^{-a^2/2\sigma_{\dot{y}}^2}\right]_a^\infty = e^{-a^2/2\sigma_{\dot{y}}^2}$$

(d) put $p = e^{-v^2/2\sigma_{\dot{y}}^2} \Rightarrow \sigma_{\dot{y}}^2 = \frac{v^2}{2\log_e(1/p)}$

(1) $\Rightarrow \frac{2k^2\pi S_1}{cm(1 + k/m\omega_0^2)} = \frac{v^2}{2\log_e(1/p)}$ $\therefore$ $c = \frac{2\pi k^2 S_1 \log_e(1/p)}{mv^2(1 + k/m\omega_0^2)}$

2. (a) The time taken for the vehicle to travel distance X is $\tau = \dfrac{X}{V}$ ——— (1)

A cycle of wavelength $\lambda = 2\pi/\gamma$ is covered in period T, where

$$T = \lambda/V = 2\pi/\omega$$

So $\omega = V\gamma$ ——— (2)

Now, the temporal spectral density and autocorrelation functions are related by the usual FT:

$$S_{yy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yy}(\tau)\, e^{-i\omega\tau}\, d\tau$$

So from (1) & (2)

$$S_{yy}(\omega = V\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yy}\left(\tau = \frac{X}{V}\right) e^{-iV\gamma\left(\frac{X}{V}\right)} \frac{1}{V}\, dX$$

$$= \frac{1}{V}\Big\{ \underbrace{\frac{1}{2\pi}\int R_{yy}(X)\, e^{-i\gamma X}\, dX}_{S_{yy}(\gamma) \text{ from question}} \Big\}$$

So $\underline{\underline{S_{yy}(\omega) = \dfrac{1}{V} S_{yy}(\gamma = \omega/V)}}$ ——— (3)

(b) $R_{yy}(X) = Ae^{-b|X|}$ $\quad -\infty < X < \infty$

Using F.T. relationship from question

$$S_{yy}(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(Ae^{-b|X|}\right) e^{-i\gamma X}\, dX$$

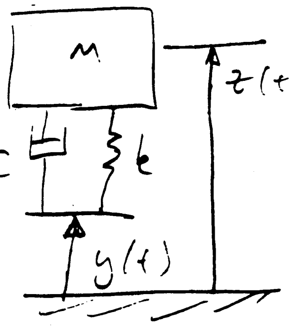
$$= \frac{A}{2\pi}\int_{-\infty}^{0} e^{(b-i\gamma)X}\, dX + \frac{A}{2\pi}\int_{0}^{\infty} e^{-(b+i\gamma)X}\, dX$$

$$= \frac{A(1-0)}{2\pi(b-i\gamma)} - \frac{A(0-1)}{2\pi(b+i\gamma)} = \frac{Ab/\pi}{b^2+\gamma^2}$$ ——— (4)

So the spectral density observed by the moving vehicle is (from (3) & (4)):

$$S_{yy}(\omega) = \frac{1}{V}\,\frac{Ab/\pi}{b^2+(\omega/V)^2} = \frac{A}{bV\pi}\left(\frac{1}{1+(\omega/bV)^2}\right) \quad \text{— (5)}$$

(c)



$$m\ddot{z} + c\dot{z} + kz = c\dot{y} + ky$$

$$\frac{Z}{Y} = \frac{1 + i2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2 + i2\zeta\omega/\omega_n}$$

$$\zeta = \frac{c}{2\sqrt{km}}, \quad \omega_n^2 = k/m$$

Dynamic tyre force $= m\ddot{z}$

$$\therefore \frac{F}{Y} = m\frac{\ddot{Z}}{Y} = -\omega^2 m\left(\frac{Z}{Y}\right) = \frac{-\omega^2 m(1+i2\zeta\omega/\omega_n)}{1-\omega^2/\omega_n^2 + i2\zeta\omega/\omega_n}$$

$$S_{ff}(\omega) = |H_{F/Y}|^2 S_{yy}(\omega)$$

$$= \left|\frac{-\omega^2 m(1+i2\zeta\omega/\omega_n)}{1-\omega^2/\omega_n^2 + i2\zeta\omega/\omega_n}\right|^2 \left[\frac{A}{bV\pi}\;\frac{1}{1+(\omega/bV)^2}\right]$$

$$= \frac{k^2A}{bV\pi}\left[\frac{(\omega/\omega_n)^4(1+4\zeta^2\omega^2/\omega_n^2)}{(1-\omega^2/\omega_n^2)^2 + 4\zeta^2\omega^2/\omega_n^2}\right]\left[\frac{1}{1+(\omega/bV)^2}\right]$$

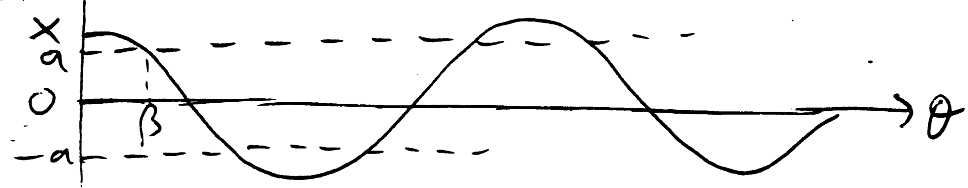
Using the M.S. Bandwidth approximation (assuming narrow band)

$$E[f^2] = \int_{-\infty}^{\infty} S_{ff}(\omega)\,d\omega \approx 2S_{ff}(\omega_n)\,\pi\zeta\omega_n$$

$$\approx 2\frac{k^2A}{bV\cancel{\pi}}\left[\frac{1}{4\zeta^{\cancel{2}}}\right]\left[\frac{1}{1+(\omega_n/bV)^2}\right]\cancel{\pi}\,\cancel{\zeta}\,\omega_n$$

$$= \frac{k^3A}{cbV}\left[\frac{1}{1+\left(\frac{k}{mb^2V^2}\right)}\right] \quad [N^2]$$

3 (a) Input: $X \cos\theta$



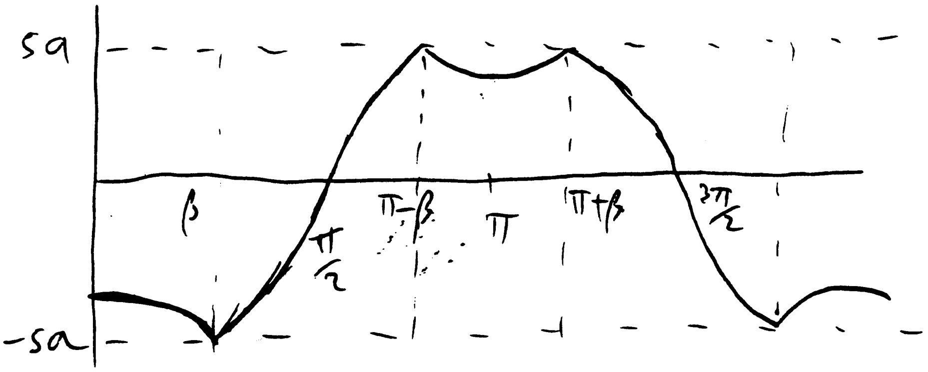
If $X < a$, output is simply $-s \times$ input, and DF $= -s$

If $X > a$, introduce β as above, where $X \cos\beta = a$.

For $0 \leq t \leq \beta$, output is shifted down by $2a$, and scaled by s

For $\beta \leq t \leq \pi/2$, output is scaled by $-s$.

This completes $1/4$ cycle, and rest of output waveform follows:



(b) So for $X > a$, output $y(\theta) = \begin{cases} s(X\cos\theta - 2a) & 0 \leq t \leq \beta \\ -sX\cos\theta & \beta \leq t \leq \pi/2 \end{cases}$

So DF $D = \dfrac{4}{X\pi}\left\{\int_0^\beta s(X\cos\theta - 2a)\cos\theta\, d\theta - \int_\beta^{\pi/2} sX\cos^2\theta\, d\theta\right\}$

$$= \frac{4s}{X\pi}\left\{\int_0^\beta \left[\frac{X}{2}(1+\cos 2\theta) - 2a\cos\theta\right]d\theta - \int_\beta^{\pi/2}\frac{X}{2}(1+\cos 2\theta)\,d\theta\right\}$$

$$= \frac{4s}{X\pi}\left\{\left[\frac{X}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - 2a\sin\theta\right]_0^\beta - \frac{X}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_\beta^{\pi/2}\right\}$$

$$= \frac{2s}{\pi}\left(\beta + \frac{\sin 2\beta}{2}\right) - \frac{8as}{X\pi}\sin\beta - \frac{2s}{\pi}\left(\frac{\pi}{2} - \beta - \frac{\sin 2\beta}{2}\right)$$

$$= \frac{2s}{\pi}\left(2\beta + 2\sin\beta\cos\beta - \frac{\pi}{2}\right) - \frac{8as}{X\pi}\sin\beta = \frac{2s}{\pi}\left[2\beta - \frac{\pi}{2} - \frac{2a}{X}\sin\beta\right]$$

where $\beta = \cos^{-1}(a/X)$

3 cont.

(c) As $X \to \infty$, $\beta \to \pi/2$ so $\sin\beta \to 0$

So $D \to \frac{2s}{\pi}\left[\pi - \frac{\pi}{2}\right] = s$

This is as expected: for very large amplitude, the negative-stiffness "kink" in $k(x)$ becomes negligible, and the system behaves like a linear stiffness s!

(d) In the describing function approximation, the equation is

$$m\ddot{x} + Dx \simeq f\cos\omega t$$

For $x = X\cos\omega t$;

$X < a \rightarrow \quad -m\omega^2 X - sX \simeq f$

$$\therefore X \simeq \frac{-f}{s+m\omega^2}$$

For $X > a$, $\quad -m\omega^2 X + \frac{2s}{\pi}\left[2\beta - \frac{\pi}{2} - \frac{2a}{X}\sin\beta\right]X \simeq f$

No explicit expression for X here, because of the X hidden inside β. Response curve would have to be computed.

4 (a) $\ddot{x} + kx - x^2 = 0$

$$\rightarrow \begin{cases}\dot{x} = y \\ \dot{y} = -kx + x^2\end{cases}$$

Stationary points $y = 0$, $kx = x^2 \rightarrow x = 0$ or k

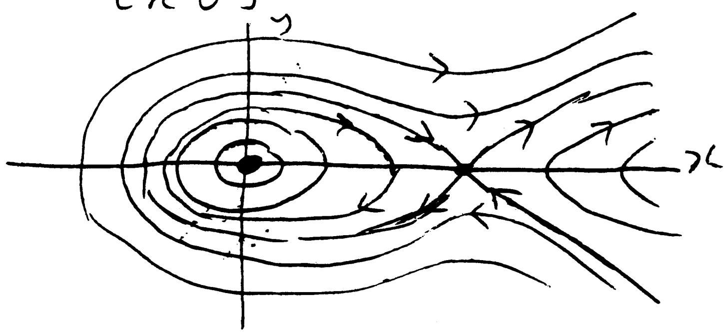
Near $(0,0)$ $\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} \simeq \begin{bmatrix}0 & 1 \\ -k & 0\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}$

Eigenvalues: $\lambda^2 + k = 0 \rightarrow \lambda = \pm i\sqrt{k} \rightarrow$ centre (stable)

Near $(k,0)$: set $x = k + \varepsilon$

Then $\begin{cases}\dot{\varepsilon} = y \\ \dot{y} = -k^2 - k\varepsilon + k^2 + 2k\varepsilon + \varepsilon^2 \simeq k\varepsilon\end{cases}$

So matrix is $\begin{bmatrix}0 & 1 \\ k & 0\end{bmatrix}$, eigenvalues $\lambda^2 - k = 0 \rightarrow$ saddle (unstable)



For potential $V(x)$, need $\dfrac{dV}{dx} = kx - x^2$

$\therefore V = \frac{1}{2}kx^2 - x^3/3 \quad (+V_0)$

This has a minimum at $x = 0$ and a maximum at $x = k$.

(b) If $k < 0$, the analysis is unchanged, except that for $(0,0)$, $\lambda^2 + k = 0 \rightarrow$ real roots, so this becomes a saddle (unstable), while for $(k,0)$, $\lambda^2 - k = 0$ gives imaginary roots so this becomes a centre

(c) $\ddot{x} + b\dot{x} + kx - x^2 = 0$, $k < 0$

$$\therefore \begin{cases}\dot{x} = y \\ \dot{y} = -by - kx + x^2\end{cases}$$

4 cont. Singular points are still the same: $y = 0$, $x = 0, k$.

Near $(0,0)$ $\begin{cases} \dot{x} = y \\ \dot{y} \simeq -by - kx \end{cases}$ $\rightarrow$ matrix $\begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}$

So eigenvalues λ satisfy $\lambda(b+\lambda) + k = 0$

$\therefore \lambda = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4k}\right)$

But $k < 0$, so $b^2 - 4k$ is always positive, $> b^2$

$\therefore$ Roots are always real, one positive, one negative.

$\therefore$ saddle point (unstable)

Near $(k, 0)$, set $x = k + \varepsilon$

Then $\begin{cases} \dot{\varepsilon} = y \\ \dot{y} = -by - k(k+\varepsilon) + (k+\varepsilon)^2 \simeq -by + k\varepsilon \end{cases}$

$\therefore$ Matrix is $\begin{bmatrix} 0 & 1 \\ k & -b \end{bmatrix}$, so eigenvalues satisfy $-\lambda(\lambda+b) - k = 0$

$\therefore \lambda = \frac{1}{2}\left(-b \pm \sqrt{b^2 + 4k}\right)$

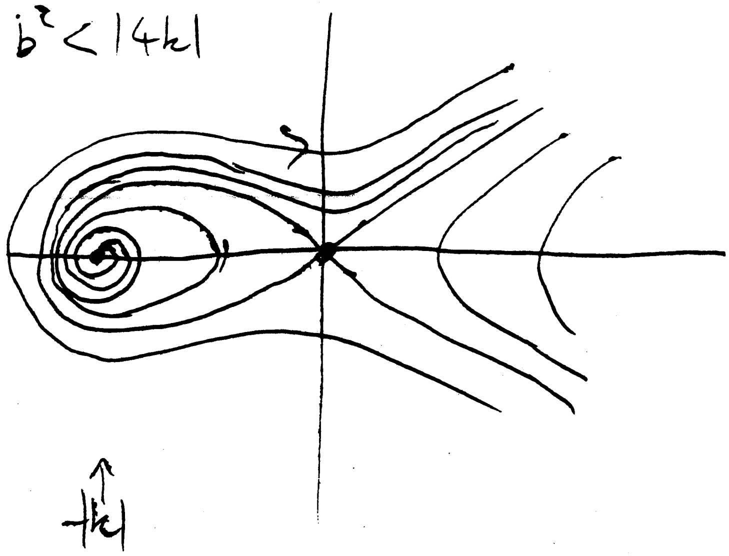
If $b^2 < |4k|$, λ's are complex with $\mathrm{Re}(\lambda) < 0$

$\therefore$ stable spiral

If $b^2 > |4k|$, two real roots both negative

$\therefore$ stable node.

$b^2 < |4k|$



$b^2 > |4k|$