Part IIB – Module 4C9 – 2005 – Solutions

1(a)

(i) $\hat{n} = \frac{1}{\sqrt{14}}(2, 1, 3)$ $\quad \sigma_{ij} = \begin{bmatrix} 3 & 4 & -8 \\ 4 & -2 & 6 \\ -8 & 6 & 5 \end{bmatrix}$

$t_i = \sigma_{ij} n_j$ (Data sheet)

$\therefore t_1 = \sigma_{1j} n_j = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$

$= \frac{1}{\sqrt{14}}\{3.2 + 4.1 + -8.3\} \Rightarrow -3.74 \text{ MPa}$

$t_2 = \sigma_{2j} n_j = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$

$= \frac{1}{\sqrt{14}}\{4.2 + -2.1 + 6.3\} \Rightarrow +6.41 \text{ MPa}$

$t_3 = \sigma_{3j} n_j = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$

$\frac{1}{\sqrt{14}}\{-8.2 + 6.1 + 5.3\} = 1.34 \text{ MPa}$

(ii) $\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$ (Data sheet)

so $\sigma'_{11} = a_{1i} a_{1j} \sigma_{ij}$

$= a_{11} a_{11} \sigma_{11} + a_{11} a_{12} \sigma_{12} + a_{13} a_{13} \sigma_{13}$ $\quad i = 1\ j = 1,2,3$

$+ a_{12} a_{11} \sigma_{21} + a_{12} a_{12} \sigma_{22} + a_{12} a_{13} \sigma_{23}$ $\quad i = 2\ j = 1,2,3$

$+ a_{13} a_{11} \sigma_{31} + a_{13} a_{12} \sigma_{32} + a_{13} a_{13} \sigma_{33}$ $\quad i = 3\ j = 1,2,3$

$\Rightarrow .866 \times .866 \times 3 + .866 \times -.354 \times 4 + .866 \times .354 \times -8$

$0.397 \Leftarrow \quad + -.354 \times .866 \times 4 + -.354 \times -.354 \times -2 + -.354 \times .354 \times 6$

$+ .354 \times .866 \times -8 + .354 \times -.354 \times 6 + .354 \times .354 \times 5$

and $\sigma'_{23} = a_{2i} a_{3j} \sigma_{ij}$

$\Rightarrow a_{21} \cancel{a_{31}} \sigma_{11} + a_{21} a_{32} \sigma_{12} + a_{21} a_{33} \sigma_{13}$ $\quad i = 1\ j = 1,2,3$

$+ a_{22} \cancel{a_{31}} \sigma_{21} + a_{22} a_{32} \sigma_{22} + a_{22} a_{33} \sigma_{23}$ $\quad i = 2$

$+ a_{23} \cancel{a_{31}} \sigma_{31} + a_{23} a_{32} \sigma_{32} + a_{23} a_{33} \sigma_{33}$ $\quad i = 3$

$\Rightarrow$ pto

$$0.5 \times .7071 \times 4 \quad + \quad 0.5 \times .7071 \times -8$$

$$+ \quad .612 \times .7071 \times -2 \quad + \quad .612 \times .7071 \times 6$$

$$+ \quad -.612 \times .7071 \times 6 \quad + \quad -.612 \times .7071 \times 5$$

$$\Rightarrow \quad -4.45 \text{ MPa}$$

Question 1 Attempts 6, average mark 15.0/20, maximum 18/20.
Only silly arithmetic errors or leaving out units marred part (a); some good answers to part (b)

Question 2 Attempts 7, average mark 16.0/20, maximum 20/20.
A straightforward torsion question which was well answered.

Question 3 Attempts 1, average mark 14.0/20, maximum 14/20.
Only one attempt which was going along the right lines.

1. (b)

(i) Let $\dot{u}^*$ be any kinematically admissible displacement rate in the direction of T, and let σ_{ij}^* be any stress state on the yield surface that could lead to $\dot{u}^*$ and strain field $\dot{\varepsilon}_{ij}^*$. Then

$$T^L \leq \frac{\int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* \, dV}{\dot{u}^*} \qquad \text{upper-bound theorem}$$

This theorem is built upon the assumption that the body is elastic-perfectly plastic and satisfies Drucker's postulates for stable solids.

(ii) From J_2-flow theory for elastic-perfectly solids:

$$\dot{\varepsilon}_{ij} = \dot{\lambda} s_{ij}$$

In simple tension, $J_2 = \frac{\sigma_y^2}{3}$ and $s_{11} = \frac{2}{3}\sigma_y$ at yielding

$$\Rightarrow \quad \dot{\varepsilon} = \dot{\lambda} \frac{2}{3} \sigma_y \quad , \quad \dot{\lambda} = \frac{3}{2} \dot{\varepsilon} \frac{1}{\sigma_y}$$

Generalising this to 3-D, we get

$$\dot{\lambda} = \frac{3}{2} \dot{\varepsilon}_e \frac{1}{\sigma_y} \qquad \text{where } \dot{\varepsilon}_e = \sqrt{\frac{2}{3} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}}$$

$$\Rightarrow \quad \boxed{\dot{\varepsilon}_{ij} = \frac{3}{2} \frac{\dot{\varepsilon}_e s_{ij}}{\sigma_y}}$$

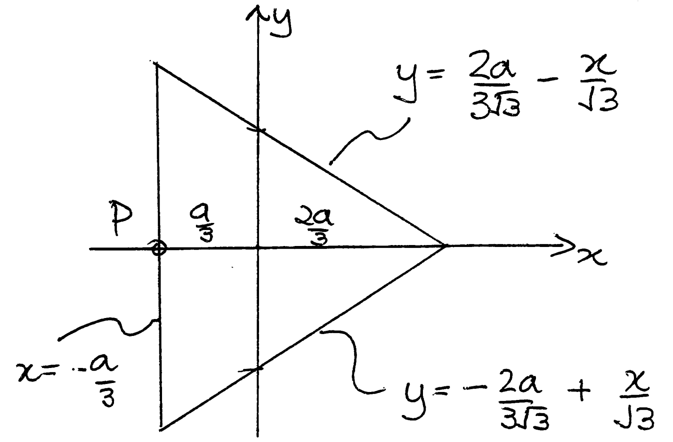
On the other hand,

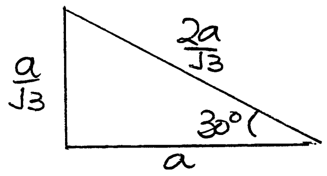
$$\int_V \sigma_{ij} \dot{\varepsilon}_{ij} \, dV = \int_V \frac{3}{2} \frac{\dot{\varepsilon}_e \sigma_{ij} s_{ij}}{\sigma_y} dV = \int_V \frac{3}{2} \frac{\dot{\varepsilon}_e}{\sigma_y} \cdot s_{ij} s_{ij} \, dV$$

But, at yielding, $\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij} = \sigma_y^2$

$$\Rightarrow \quad \int_V \sigma_{ij} \dot{\varepsilon}_{ij} \, dV = \int \frac{3}{2} \frac{\dot{\varepsilon}_e}{\sigma_y} \cdot \frac{2}{3} \sigma_y^2 \, dV \; \boxed{= \sigma_y \int_V \dot{\varepsilon}_e \, dV}$$

2. a necessary condition is that $\phi = 0$ around boundary of section.





So a possible Prandtl function is

$$c'\left(x+\frac{a}{3}\right)\left(y+\frac{x}{\sqrt{3}}-\frac{2a}{3\sqrt{3}}\right)\left(y-\frac{x}{\sqrt{3}}+\frac{2a}{3\sqrt{3}}\right)=0$$

$$\Rightarrow c'\left(x+\frac{a}{3}\right)\left(y+\left(\frac{x}{\sqrt{3}}-\frac{2a}{3\sqrt{3}}\right)\right)\left(y-\left(\frac{x}{\sqrt{3}}-\frac{2a}{3\sqrt{3}}\right)\right)=0$$

$$c'\left(x+\frac{a}{3}\right)\left\{y^2-\frac{x^2}{3}+\frac{4ax}{3.3}-\frac{4a^2}{9.3}\right\}=0$$

$$c'\left(xy^2-\frac{x^3}{3}+\frac{4ax^2}{9}-\cancel{\frac{4a^2x}{9}}+\frac{ay^2}{3}\right.$$
$$\left.-\frac{ax^2}{9}+\cancel{\frac{4a^2x}{9}}-\frac{4a^3}{3.9.3}\right)$$

$$\Rightarrow c'\left\{-\frac{x^3}{3}+xy^2+\frac{ax^2}{3}+\frac{ay^2}{3}-\frac{4a^3}{3.9.3}\right\}$$

$$\text{i.e. } \phi = \frac{ca}{3}\left\{(x^2+y^2)-\frac{1}{a}(x^3-3xy^2)-\frac{4a^2}{27}\right\}$$

$$\text{or } \phi = c\left\{(x^2+y^2)-\frac{1}{a}(x^3-3xy^2)-\frac{4a^2}{27}\right\}$$

i.e. $k = \frac{4}{27}$

$$\phi = c\left\{(x^2+y^2)-\frac{1}{a}(x^3-3xy^2)-\frac{4}{27}a^2\right\}$$

But also $\nabla^2\phi = -2G\alpha$ (Data sheet)

$$\frac{\partial\phi}{\partial x} = C\left\{2x - \frac{3x^2}{a} + \frac{3y^2}{a}\right\}; \quad \frac{\partial^2\phi}{\partial x^2} = 2C - \frac{6Cx}{a}$$

$$\frac{\partial\phi}{\partial y} = C\left\{2y + \frac{6xy}{a}\right\}; \quad \frac{\partial^2\phi}{\partial y^2} = 2C + \frac{6Cx}{a}$$

$$\therefore \nabla^2\phi = 4C = -2G\alpha \quad \text{i.e. } \underline{C = -G\alpha/2}$$

(a) $$\tau_{zy} = -\frac{\partial\phi}{\partial x} = \frac{G\alpha}{2}\left\{2x - \frac{3x^2}{a} + \frac{3y^2}{a}\right\}$$

So at mid-point of sides, eg when $x = -a/3$, $y = 0$

$$\tau_{zy} = \frac{G\alpha}{2}\left\{-\frac{2a}{3} - \frac{3}{a}\cdot\frac{a^2}{9} + 0\right\}$$

$$|\tau_{zy}| = \frac{G\alpha a}{2} \Rightarrow \tfrac{1}{2}\times 80\times 10^9 \times 5\times\pi/180 \times 0.15 \Rightarrow \underline{524\text{ MPa}}$$

(b) If warping function is $\psi(x,y)$ then $\nabla^2\psi = 0$

and $\tau_{xz} = \alpha G\left\{\frac{\partial\psi}{\partial x} - y\right\}$ and $\tau_{yz} = \alpha G\left\{\frac{\partial\psi}{\partial y} + x\right\}$

$$\psi = B\{y^3 - 3x^2y\} \quad \frac{\partial\psi}{\partial x} = -6Bxy \quad \frac{\partial^2\psi}{\partial x^2} = -6By$$

$$\frac{\partial\psi}{\partial y} = 3By^2 - 3Bx^2 \quad \frac{\partial^2\psi}{\partial y^2} = 6By$$

$$\therefore \nabla^2\psi = 0$$

and $\tau_{xz} = \alpha G\{-6Bxy - y\}$; $\tau_{yz} = \alpha G\{3By^2 - 3Bx^2 + x\}$

But $\tau_{xz} = \frac{\partial\phi}{\partial y} = \alpha G\left\{-\frac{3xy}{a} - y\right\}$

and $\tau_{yz} = -\frac{\partial\phi}{\partial x} = \alpha G\left\{x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right\}$

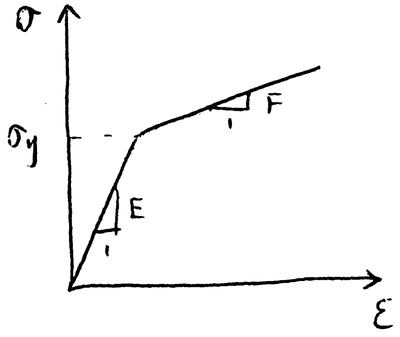
Comparing coefficients $6B = 3/a$ i.e. $\underline{B = 1/2a}$

3. (a) $\sigma = \sigma_y$ in uniaxial tension is generalised to 3-D as:

$$\sigma_e = \sigma_y \qquad \text{(Mises criterion)}$$

where $\sigma_e = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$ is the equivalent stress, and s_{ij} is the deviatoric stress. Here, s_{ij} rather than σ_{ij} itself is used because it has been assumed that hydrostatic pressure does not cause yielding; the equivalent stress is defined such that $\sigma_e = \sigma$ in 1-D. Also, it is assumed that the material is initially isotropic, so that its yielding only depends on stress invariants.

(b)



$$\varepsilon = \begin{cases} \sigma / E & \text{if } \sigma \le \sigma_y \\ \dfrac{\sigma_y}{E} + \dfrac{\sigma - \sigma_y}{F} & \text{if } \sigma \ge \sigma_y \end{cases}$$

When $\sigma \le \sigma_y$, $\quad E_s = \frac{\sigma}{\varepsilon} = E, \quad E_t = \frac{d\sigma}{d\varepsilon} = E$

When $\sigma \ge \sigma_y$

$$\frac{1}{E_s} = \frac{\varepsilon}{\sigma} = \frac{\sigma_y}{\sigma}\left(\frac{1}{E} - \frac{1}{F}\right) + \frac{1}{F} \quad \Rightarrow \quad E_s = \left[\frac{1}{F} + \frac{\sigma_y}{\sigma}\left(\frac{1}{E} - \frac{1}{F}\right)\right]^{-1}$$

$$\frac{1}{E_t} = \frac{d\varepsilon}{d\sigma} = \frac{1}{F} \qquad \Rightarrow \quad E_t = F$$

(c) In $(\sigma_{zz}, \sigma_{\theta z})$ plane, $\quad J_2 = \frac{1}{3}\left(\sigma_{zz}^2 + 3\sigma_{\theta z}^2\right)$

$$\Rightarrow \sigma_e = \sqrt{3 J_2} = \sqrt{\sigma_{zz}^2 + 3\sigma_{\theta z}^2}, \qquad \dot{J}_2 = \frac{2}{3}\left(\sigma_{zz}\dot{\sigma}_{zz} + 3\sigma_{\theta z}\dot{\sigma}_{\theta z}\right)$$

J_2- flow theory then gives

$$\dot{\varepsilon}_{zz} = \frac{\dot{\sigma}_{zz}}{E} + \frac{1}{h} s_{zz} \dot{J}_2, \qquad \dot{\varepsilon}_{\theta z} = \frac{\dot{\sigma}_{\theta z}}{2G} + \frac{1}{h} \sigma_{\theta z} \dot{J}_2$$

where

$$s_{zz} = \frac{2}{3}\sigma_{zz}$$

$$\frac{1}{h} = \frac{9}{4\sigma_e^2}\left(\frac{1}{E_t} - \frac{1}{E}\right) = \frac{9}{4\sigma_e^2}\left(\frac{1}{F} - \frac{1}{E}\right)$$

3. (c) continued

$$\Rightarrow \dot{\varepsilon}_{zz} = \frac{\dot{\sigma}_{zz}}{E} + \frac{9}{4\sigma_e^2}\left(\frac{1}{F}-\frac{1}{E}\right)\frac{2}{3}\sigma_{zz}\cdot\frac{2}{3}\left(\sigma_{zz}\dot{\sigma}_{zz}+3\sigma_{\theta z}\dot{\sigma}_{\theta z}\right)$$

$$= \frac{\dot{\sigma}_{zz}}{E} + \left(\frac{1}{F}-\frac{1}{E}\right)\frac{\sigma_{zz}\left(\sigma_{zz}\dot{\sigma}_{zz}+3\sigma_{\theta z}\dot{\sigma}_{\theta z}\right)}{\sigma_{zz}^2+3\sigma_{\theta z}^2}$$

$$\dot{\varepsilon}_{\theta z} = \frac{\dot{\gamma}_{\theta z}}{2} = \frac{\dot{\sigma}_{\theta z}}{2G} + \frac{3}{2}\left(\frac{1}{F}-\frac{1}{E}\right)\frac{\sigma_{\theta z}\left(\sigma_{zz}\dot{\sigma}_{zz}+3\sigma_{\theta z}\dot{\sigma}_{\theta z}\right)}{\sigma_{zz}^2+3\sigma_{\theta z}^2}$$

(d) $\sigma_{zz}^2 = 3\sigma_{\theta z}^2$ maintained throughout loading

$\sigma_{zz} = \sqrt{3}\,\sigma_{\theta z}$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} + \left(\frac{1}{F}-\frac{1}{E}\right)\int_0^{\sigma_{zz}} \frac{\sigma_{zz}^2\,d\sigma_{zz} + \sigma_{zz}^2\cdot\frac{(\sqrt{3})^2}{3}d\sigma_{zz}}{2\sigma_{zz}^2}$$

$$= \frac{\sigma_{zz}}{E} + \left(\frac{1}{F}-\frac{1}{E}\right)\sigma_{zz}$$

$$= \frac{\sigma_{zz}}{F}$$

$\therefore$ At $\sigma_{zz} = \sigma_y$, $\boxed{\varepsilon_{zz} = \sigma_y / F}$

JAW, TJL
June 05