4C15 2004/05 – CRIB
Dr Ashwin Seshia (aas41)

(a) (i)

$$k = \frac{\pi^4}{6} EH\left(\frac{W}{L}\right)^3$$

$$= \frac{\pi^4}{6} \times 160\times10^9 \times 3\times10^{-6} \times \left(\frac{1.5}{150}\right)^3$$

$$= 7.80 \text{ N/m}$$

(ii)

$$b = \frac{96 \times 1.8\times10^{-5} \times 50\times10^{-6}}{\pi^4} \times \left(\frac{3}{1}\right)^3$$

$$= 2.4\times10^{-8} \text{ kg/s}$$

(b) $PE_{max} = KE_{max}$ / Origin at center of b

$$\frac{E}{2}\int_{-H/2}^{+H/2}\int_{-W/2}^{+W/2}\int_{-L/2}^{+L/2}\left(y\frac{d^2u}{dx^2}\right)^2 dx\,dy\,dz = \frac{\omega^2}{2}\left[WH\int \rho\, u^2(x)\,dx\right] + \frac{1}{2} M_{actuator}\cdot\omega^2 u^2(0)$$

$$\frac{E}{2}\cdot H\left[\frac{2}{3}\left(\frac{W}{2}\right)^3\right]\left(\frac{2\pi}{L}\right)^4\frac{c^2}{4}\int_{-L/2}^{+L/2}\cos^2\left(\frac{2\pi x}{L}\right)dx = \frac{\omega^2}{2}\rho WH\left[\int_{-L/2}^{+L/2}u^2(x) + L_{actuator}\times\right]$$

$$L_{actuator} = \tfrac{1}{3}L$$

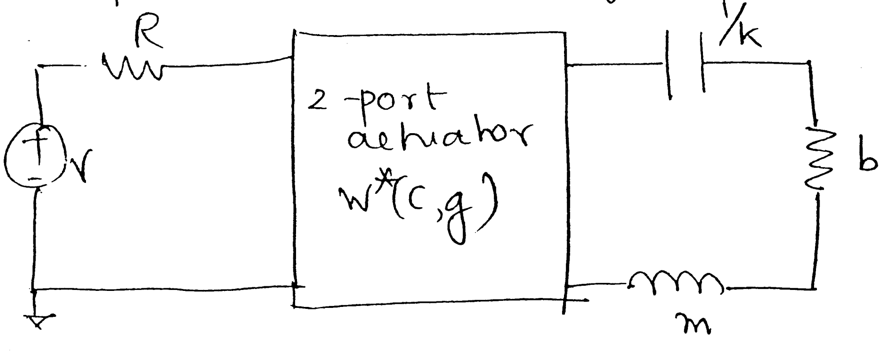
$$\frac{E}{2}H\cdot\frac{W^3}{12}\left(\frac{2\pi}{L}\right)^4\frac{c^2}{4}\cdot\frac{L}{2} = \frac{\omega^2\rho}{2}WHc^2\left(\frac{3L}{2}+\frac{L}{3}\right)$$

$$\therefore\ \omega^2 = \frac{E}{\rho}\,\frac{W^2\pi^4}{11\,L^4}$$; gives the resonant frequency

(c) Lumped mass, $m = k/\omega^2$

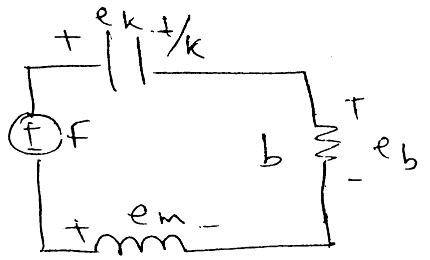
$$m = \frac{\pi^4 EH\left(\frac{W}{L}\right)^3}{\frac{\pi^4}{11}\cdot\frac{E}{\rho}\,\frac{W^2}{L^4}} = \frac{11}{6}\rho(LWH)$$

A lumped element equivalent model is



electrical circuit mechanical circuit.

The mechanical circuit



$$F - e_k - e_b - e_m = 0$$

$$F - kx - b\dot{x} - m\ddot{x} = 0$$

$$\frac{X(s)}{F(s)} = \frac{1}{k + bs + ms^2}$$

where $k = 7.8\ N/m$
$m = 2.88 \times 10^{-12}\ kg$
$b = 2.4 \times 10^{-8}\ kg/s$

(e) small signal AC

$$Q = \frac{\sqrt{km}}{b} = \frac{\sqrt{7.8 \times 2.9 \times 10^{-12}}}{2.4 \times 10^{-8}} = 198$$

$V_{ac} = 50\ mV$, $V_{DC} = 5\ V$

$$F = \frac{1}{2} \frac{\epsilon_0 A}{g^2} \cdot (2 V_{ac} \cdot V_{DC})$$

$$\simeq \frac{\frac{1}{2} \times 8.85 \times 10^{-12} \times 50 \times 3 \times 2 \times 50 \times 5 \times 10^{-3}}{1 \times 1}$$

X_{ac} = displacement at resonance

$$\simeq \frac{F}{k} \cdot Q \quad ; \quad k = 7.8\ N/m,\ Q = 198$$

$$\simeq 8.4\ nm$$

∴ our assumption on small displacement is valid.

(a) C = capacitance between membrane and the ~~sub~~ electrode attached to substra

$$= \int_{-y_0}^{y_0}\int_{-x_0}^{x_0} \frac{\varepsilon_0}{(g - w(x,y))}\, dx\, dy$$

where
$x_0 = 10\,\mu m$
$y_0 = 10\,\mu$
$L = 1000$

$$C = \int_{-y_0}^{y_0}\int_{-x_0}^{x_0} \frac{\varepsilon_0 \; dx\, dy}{\left(g - \frac{c_1}{4}\left(1+\cos\frac{2\pi x}{L}\right)\left(1+\cos\frac{2\pi y}{L}\right)\right)}$$

$$C = \frac{\varepsilon_0}{g}\int_{-y_0}^{y_0}\int_{-x_0}^{+x_0} \frac{dx\, dy}{\left(1 - \frac{c_1}{4g}\left(1+1-\left(\frac{2\pi x}{L}\right)^2\frac{1}{2}\right)\left(1+1-\left(\frac{2\pi y}{L}\right)^2\frac{1}{2}\right)\right)}$$

using Taylor expansion for $\cos\theta$ noting $x_0 \ll L, y_0 \ll$

$$C = \frac{\varepsilon_0}{g}\int_{-y_0}^{+y_0}\int_{-x_0}^{+x_0} \frac{dx\, dy}{\left(1 - \frac{c_1}{g}\left(1-\frac{1}{2}\left(\frac{\pi x}{L}\right)^2\right)\left(1-\left(\frac{\pi y}{L}\right)^2\frac{1}{2}\right)\right)}$$

In the limit $x_0 \ll L, y_0 \ll L$ this expression can be simplified to $\rightarrow$ $(x_0 = 10, y_0 = 10, L = 1000)$

$$C = \frac{\varepsilon_0 4 x_0 y_0}{g}\left(\frac{c_1}{g}\right) + \frac{\varepsilon_0 4 x_0 y_0}{g}$$

$$C = \frac{4\varepsilon_0 x_0 y_0}{g} \cdot \frac{6P(1-\nu^2)L^4}{\pi^4 E H^3 \cdot g} + \frac{\varepsilon_0 4 x_0 y_0}{g}$$

(b) Scale factor of the device

$$\frac{\Delta C}{\Delta P} = \frac{d(C - C_0)}{dP} = \frac{24\,\varepsilon_0 x_0 y_0 (1-\nu^2)L^4}{\pi^4 g^2 E H^3}$$

For an applied pressure load of 500 Pa

$$\frac{\Delta C}{P} = \text{scale factor}$$

$$\frac{\Delta C}{P} = \frac{24 \times 8.85 \times 10^{-12} \times 10 \times 10 \times 10^{-12} (1 - 0.27^2)}{160 \times 10^9 \times \pi^4 \times 5 \times 10^{-6} \times 5} \times \left(\frac{1000}{5}\right)^3 \times 100$$

$$\frac{\Delta C}{P} = 4.04 \times 10^{-19} \quad F/Pa$$

(c) Damping constant, b

$$b = \frac{96\, \eta\, L^4}{\pi^4 g^3}$$

$$= \frac{96 \times 1.8 \times 10^{-5}}{\pi^4} \times \left(\frac{1000}{5}\right)^3 \times 1000 \times 10^{-6}$$

$$= 0.14 \ \text{kg/s}$$

$\therefore$ (a) $\dfrac{d^2 u_x}{dz^2} = \dfrac{\sigma_w E_x}{\eta L_D} e^{-z/L_D}$, $z = 0$ at the interface

$u = u_0 (1 - e^{-z/L_D})$ noting that

$u(0) = 0$ (no-slip condition).

$\dfrac{du}{dz} = \dfrac{u_0 e^{-z/L_D}}{L_D}$ and $\dfrac{d^2 u}{dz^2} = -\dfrac{u_0}{L_D^2} e^{-z/L_D}$.

or $u_0 = \dfrac{-\sigma_w E_x L_D}{\eta}$

(b) $\sigma_w = \dfrac{\phi_w \varepsilon}{L_D} = \dfrac{80 \times 10^{-3} \times 80 \times 8.85 \times 10^{-12}}{0.5 \times 10^{-9}}$

$\sigma_w = 0.11 \ C/m^2$

(c) $u_0 = \dfrac{0.11 \times 500 \times 10^2 \times 0.5 \times 10^{-9}}{1.5 \times 10^{-3}} = 1.83 \times 10^{-3}$ m/s

Q = volumetric flow rate

$= A u_0$

$= 30 \times 50 \times 10^{-12} \times 1.83 \times 10^{-3} \ m^3/s$

$= 2.74 \times 10^{-12} \ m^3/s$.

t_1 = time taken to traverse 1cm

$= \dfrac{1 \text{ cm}}{1.83 \times 10^{-3}} = 5.46$ secs.

(d) drift velocity due to electrophoresis

$= \mu_{ep} \cdot E_x$.

$\therefore t_2$ = time for 2 biomolecules to separate by 10μm.

$$t_2 = \frac{10^{-5} \times 1.5\times10^{-3}}{2\times10^{5} \times 4\times1.6\times10^{-19}\times 500\times100}$$

$$t_2 = 2.34 \text{ secs.}$$

distance travelled by the bulk solution

$= t_2 . U_0$

$= 2.34 \times 1.83 \times 10^{-3}$

$= 4.29$ mm.

(e) width of the band after separation by 10μm

$\simeq \sqrt{D . t_2}$

$\simeq \sqrt{10\,\mu m^2/s . 2.34 secs}$

$= 4.8\ \mu m$. (close to the separation i.e bands just beginning to separate).

Band size $= \sqrt{D . t}$

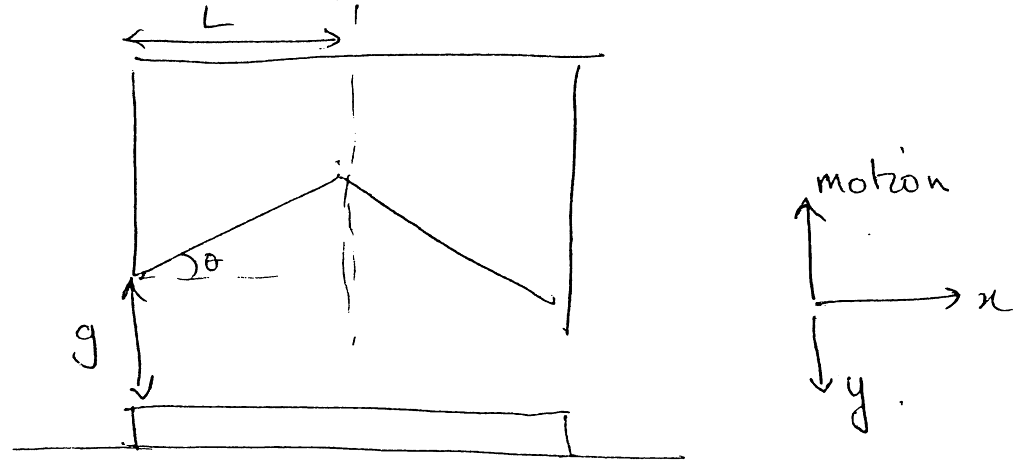
$= \sqrt{D . L / U_0}$

$= \sqrt{D . L \eta / (\sigma_w . E_x L_D)}$.

∴ to obtain the sharpest bands use short columns, large electric fields and relatively low ionic strength buffers to achieve large Debye length.

Q4

(a) $W = \frac{1}{2} C V^2$



First calculate the capacitance of element size dx along the length

$$dC = \frac{\varepsilon_0 w \, dx}{(g + x\tan\theta - y)}$$

$$C = 2\int_0^L \frac{\varepsilon_0 w \, dx}{(g + x\tan\theta - y)} = 2\varepsilon_0 w \int_0^L \frac{dx}{(g - y + x\tan\theta)}$$

$$= \frac{2\varepsilon_0 w}{(g-y)} \int_0^L \frac{dx}{\left[1 + \left(\frac{\tan\theta}{g-y}\right) x\right]}$$

$$= \frac{2\varepsilon_0 w}{(g-y)} \, \frac{\ln\left(1 + \frac{\cancel{x} L\tan\theta}{g-y}\right)}{\frac{\tan\theta}{(g-y)}}$$

$$C = \frac{2\varepsilon_0 w}{\tan\theta} \ln\left(1 + \frac{L\tan\theta}{g-y}\right)$$

$$F = \frac{\partial}{\partial y} \frac{1}{2} C V^2 = \frac{\cancel{2}\varepsilon_0 w V^2}{\tan\theta} \frac{\partial}{\partial y} \ln\left(1 + \frac{L\tan\theta}{g-y}\right)$$

$$F = -\frac{\varepsilon_0 w V^2}{\tan\theta} \frac{1}{\left(1 + \frac{L\tan\theta}{g-y}\right)} \frac{(L\tan\theta)}{(g-y)^2}$$

$$F = \frac{-\varepsilon_0 w V^2 L}{(g-y)(g-y+L\tan\theta)}$$

Static displacement $= t/k$ *

(b) The pull-in instability $-\partial F/\partial y = k$ and $F_{net} = 0$

or $\dfrac{\varepsilon_o w L V^2}{(g-y)(g-y+L\tan\theta)}\left[\dfrac{1}{(g-y)} + \dfrac{1}{(g-y+L\tan\theta)}\right] = k$. — (1)

and $F_{net} = 0$

$\Rightarrow \left[-1 + \dfrac{y}{(g-y)} + \dfrac{y}{(g-y+L\tan\theta)}\right] = 0$

$-(g-y+L\tan\theta)(g-y) + y(g-y) + y(g-y+L\tan\theta) = 0.$

$3y^2 + y(-4g - 2L\tan\theta) + (g^2 - gL\tan\theta) = 0$

$$y = \frac{4g + 2L\tan\theta \pm \sqrt{(4g+2L\tan\theta)^2 - 4(g^2 - gL\tan\theta)(3)}}{6}.$$

$$y = \frac{2g + L\tan\theta \pm \sqrt{g^2 + L^2\tan^2\theta + 7gL\tan\theta}}{3}$$

Choose the negative root as makes physica sense i.e. $y < g$. is required.

In the limit $\theta \to 0$, $y \to \dfrac{2g - g}{3}$ or $y \to g/3$

which is the same as that for the parallel plate case

(c) The geometry shown minimizes contac area between the plates. Hence less susceptible to stiction, a common failure mechanism in these devices.