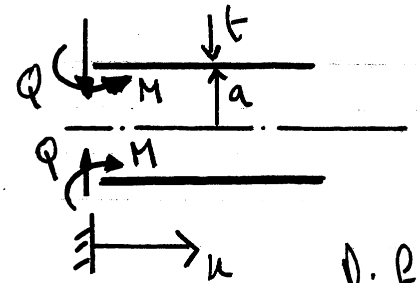
2005 IIB 4D2 Dr K Seffen kas14@eng.cam.ac.uk

# Qu 1 4D2 2004-5

(a)




Q and M, positive, as drawn. The governing equation of radial deformation is

$$D\frac{d^4w}{dx^4} + \frac{Et}{a^2}w = p$$

D: flexural rigidity $= Et^3/12(1-\nu^2)$
p: transverse pressure
w: radial (outwards) deflexion.

(b) Here $p = 0$; define $\beta^4 = 3(1-\nu^2)/a^2t^2 \Rightarrow \underline{\underline{\dfrac{d^4w}{dx^4} + 4\beta^4 w = 0}}$

which has general solution $w = A_1e^{\beta x}\cos\beta x + A_2e^{\beta x}\sin\beta x + A_3e^{-\beta x}\cos\beta x + A_4e^{-\beta x}\sin\beta x$.

The terms in $e^{\beta x}$ must attenuate as $x \to \infty \Rightarrow A_1$ and $A_2 = 0$

$$\Rightarrow \quad w = e^{-\beta x}\left[A_3\cos\beta x + A_4\sin\beta x\right]$$

$$\frac{dw}{dx} = \beta e^{-\beta x}\left[-A_3(\cos\beta x + \sin\beta x) + A_4(\cos\beta x - \sin\beta x)\right]$$

$$d^2w/dx^2 = \beta^2 e^{-\beta x}\left[2A_3\sin\beta x - 2A_4\cos\beta x\right]$$

$$d^3w/dx^3 = \beta^3 e^{-\beta x}\left[2A_3(\cos\beta x - \sin\beta x) + 2A_4(\cos\beta x + \sin\beta x)\right]$$

As defined, $M = -D\,d^2w/dx^2\big|_{x=0} \Rightarrow M = -D\beta^2\left[2A_3.0 - 2.A_4.\right]$

$$\Rightarrow A_4 = M/2\beta^2 D$$

$Q = dM/dx = -D\,d^3w/dx^3\big|_{x=0}$

$$\Rightarrow Q = -D\beta^3\left[2A_3.1 + 2A_4.1\right]$$

$$\Rightarrow \quad -Q/2D\beta^3 = A_3 + A_4 \Rightarrow A_3 = -Q/2D\beta^3 - M/2\beta^2 D$$

$$\Rightarrow \quad w = e^{-\beta x}\left[\cos\beta x.\left[-Q/2\beta^3 D - M/2\beta^2 D\right] + \sin\beta x.\left[M/2\beta^2 D\right]\right]$$

or

$$w = \frac{e^{-\beta x}}{2\beta^2 D}\left[-Q/\beta.\cos\beta x - M\cos\beta x + M.\sin\beta x\right]$$

$$\Rightarrow \quad \boxed{w = \frac{e^{-\beta x}}{2\beta^2 D}\left[M(\sin\beta x - \cos\beta x) - Q/\beta\cos\beta x\right]}$$

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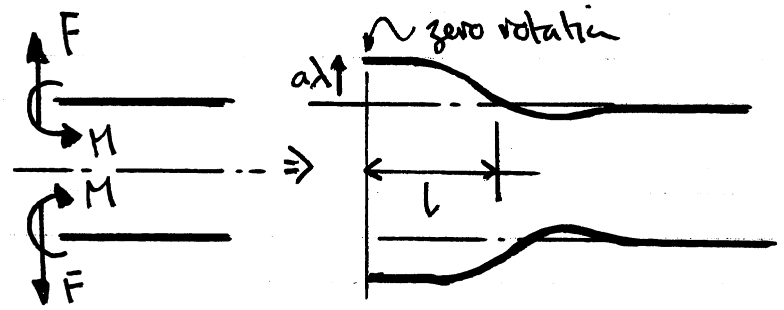
Correspondingly $\frac{dw}{dx} = \frac{e^{-\beta x}}{2\beta D}\left[(Q/\beta + M)(\cos\beta x + \sin\beta x) + M(\cos\beta x - \sin\beta x)\right]$

$$\Rightarrow \boxed{\frac{dw}{dx} = \frac{e^{-\beta x}}{2\beta D}\left[ Q/\beta\,[\cos\beta x + \sin\beta x] + M \,.\, 2\cos\beta x\right]}$$

$\Rightarrow$ radial deflexion $(x=0) = (M + Q/\beta)/2\beta^2 D \downarrow$
rotation $(x=0) = (2M + Q/\beta)/2\beta D$

(c). The heating and joining process is tantamount to the following mechanical set-up.




F is in opposite sense to Q : for zero rotation at the $x=0$ end

$\Rightarrow \quad 2M - F/\beta = 0$ (from rotation) $\Rightarrow$ $\underline{F = 2M\beta}$.

The deflexion (upwards) $= 3M/2\beta^2 D \uparrow = a\lambda \Rightarrow \underline{M = 2\beta^2 D a\lambda/3}$

For deflexion along $x$, then $w = \frac{e^{-\beta x}}{2\beta^2 D}\left[M(\sin\beta x - \cos\beta x) + \underbrace{\frac{2M\beta}{\beta}}_{F=-Q}\cos\beta x\right]$

$\Rightarrow \quad w \propto [\sin\beta x + \cos\beta x]$ ; when $w = 0$, $x = l$

$\Rightarrow \quad \tan\beta \underset{\uparrow l}{x} = -1$ or $\beta \underset{\uparrow l}{x} = 3\pi/4$

$\Rightarrow \quad l = 3\pi/4\beta = \frac{3\pi}{4}\cdot 1/3^{1/4}(1-\nu^2)^{1/4}\cdot(a^2t^2)^{1/4}$

$\Rightarrow \quad l = \frac{3^{3/4}\pi}{4(1-\nu^2)^{1/4}}\cdot\sqrt{at} \simeq \underline{1.8\sqrt{at}}$

③

# Qu2: 4D2 2004-5.

$w = \frac{B}{2}\left[x^2 - \nu y^2\right] + Cxy$ : Cartesian system $\Rightarrow$ $K_x = -\partial^2 w/\partial x^2$

$K_y = -\partial^2 w/\partial y^2$

$K_{xy} = -\partial^2 w/\partial x \partial y$

$\therefore \boxed{K_x = -B \; ; \; K_y = +\nu B \quad ; \; K_{xy} = -C}$

$M_x = D[K_x + \nu K_y]$; $M_y = D[K_y + \nu K_x]$; $M_{xy} = D(1-\nu)K_{xy}$

$\Rightarrow \quad M_x = -\underline{BD[1-\nu^2]}$ ; $M_y = D[\underbrace{\cancel{\nu B - \nu B}}_{0}]$ ; $M_{xy} = -\underline{D(1-\nu)C}$

Thus $M_x$ and $M_{xy}$ are constants ; $M_y = 0$ (internally).

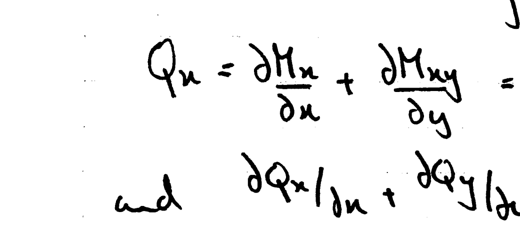
$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = 0$ ; as $Q_y$ [moment eqlm of an element]

and $\partial Q_x/\partial x + \partial Q_y/\partial y = -p \Rightarrow p = 0$ [force eqlm of element].

$\therefore$ no external pressure of distributed shear forces/length. However, since we deal with a rectangular plate, must also consider Kelvin's Edge resultant. For example, in the $x$-edge ($x = \pm a/2$)

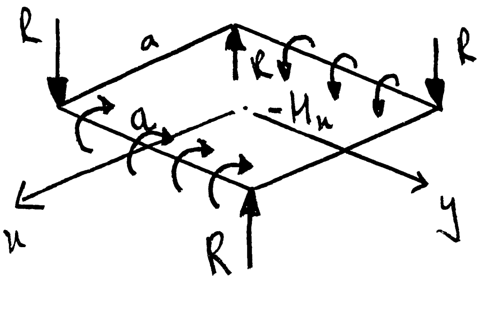
$$V_x = Q_x + \partial M_{xy}/\partial y = 0 \quad \text{(as is } V_y \text{ on } y\text{-edges).}$$

But cannot discount corner forces, $R = 2M_{xy} = -2DC(1-\nu)$.




reactn { +ve dirn shown in +ve x-y quadrant

$\therefore$ The forces <u>applied</u> to the corners must be in the opposite sense (and consistent with deflexion profile). In addition a moment is applied to the '$x$-edges to generate $M_x$ internally




The <u>total</u> moment $M = -aM_x = aBD(1-\nu^2)$

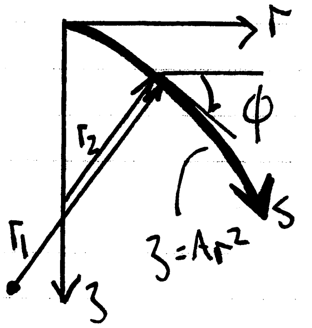
Given $D = \frac{Et^3}{12(1-\nu^2)} \Rightarrow M = a.B.Et^3/12$

or $B = 12M/Eat^3$

The torque/couple due to R is $aR = T$, say

$\Rightarrow \quad C = 6T(1+\nu)/Eat^3$

# Qu 3: 4D2 2004-5




$$\left.\begin{array}{l} 1/r_1 = \dfrac{\partial \phi}{\partial s} \; ; \; 1/r_2 = 1/r \; dz/ds \; : \; \delta s \neq \delta r \quad \text{FUNDAMENTAL} \\ \sin\phi = dz/ds \; ; \; \cos\phi = dr/ds \\ \Rightarrow \tan\phi = dz/dr \end{array}\right\}$$

$\rightarrow$ background geometry.

a) Membrane hypothesis : bookwork. If loading + boundary conditions permit development of mainly in-plane forces, either tensile or compressive, this manifests as the membrane hypothesis in terms of a solution approach.

b) $\tan\phi = dz/dr = 2Ar \quad \Rightarrow \quad r = \tan\phi / 2A$

$1/r_1$ (principal) $= \partial\phi/\partial s = d\phi/dr \cdot dr/ds \; ; \; dr/d\phi = 1/2A \cdot \sec^2\phi$.

$$\Rightarrow 1/r_1 = \frac{2A}{\sec^2\phi} \cdot \cos\phi = \underline{2A\cos^3\phi}$$

$$1/r_2 = 1/r \; dz/ds = \frac{2A}{\tan\phi} \cdot \sin\phi = \underline{2A\cos\phi}$$

With internal pressure, use equilibrium of a cap:

$$2\pi r N_\phi \cdot \sin\phi = p \cdot \pi r^2 \Rightarrow N_\phi = pr/2\sin\phi$$

$$\Rightarrow N_\phi = \frac{p}{2\sin\phi} \cdot \frac{\tan\phi}{2A} = \underline{p/4A\cos\phi}$$

Vertical eqlm of an element $\Rightarrow N_\phi/r_1 + N_\theta/r_2 = p$

$$\Rightarrow \frac{p}{2\cancel{4A\cos\phi}} \cdot \cancel{2A}\cos^{2}\phi + N_\theta \cdot 2A\cos\phi = p$$

$$\underline{N_\theta = \frac{1}{2A\cos\phi} \cdot \left[ p - p\cos^2\phi/2 \right]}$$

For small $\phi$, $\cos\phi \simeq 1 \Rightarrow N_\phi = p/4A$ ; $N_\theta = p/2A\,[1 - 1/2] = p/4A$

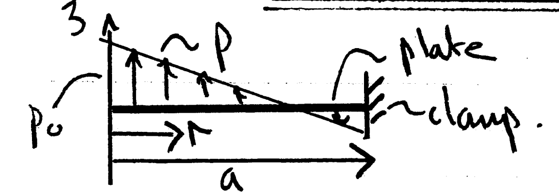
$$\Rightarrow \underline{N_\phi \simeq N_\theta \text{ for small } \phi}$$

c) $N_\theta = p\, dA/ds$ ; this is Zick's law. For small $\phi \sim N_\theta/p \simeq \dfrac{1}{4A}$

$$\frac{dA}{ds} \simeq \frac{dA}{d\phi} \cdot \frac{d\phi}{ds} \; : \; A = 1/2\, r_2^2 \phi \simeq \phi/8A^2 \; ; \; s = r_2\phi \simeq \phi/2A$$

$$\Rightarrow \frac{dA}{ds} \simeq \frac{1}{8A^2} \cdot 2A = 1/4A \quad \text{QED}$$

# Qu 4: 4D2 2004-5




standard circular plate problem.

linear: above/below plate, depending on β.

a) $\frac{D}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right] = p = p_0\left[1 - r/\beta a\right]$  $D \equiv$ flexural rigidity

$\times \frac{r}{D}$ (b) $\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right] = \frac{p_0}{D}\left[r - r^2/\beta a\right]$

$\int, \div r$ $\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right) = \frac{p_0}{D}\left[\frac{r}{2} - r^2/3\beta a\right] + C_1$  constant; 0 due to avoiding infinite shear at centre

$\int, \times r$ $d/dr\left(r\,dw/dr\right) = \frac{p_0}{D}\left[\frac{r^3}{4} - \frac{r^4}{9\beta a}\right] + C_2 r$  constant

$\int, \div r$ $dw/dr = p_0/D\cdot\left[r^3/16 - r^4/45\beta a\right] + C_2\frac{r}{2} + C_3/r$

$\int$ $w = p_0/D\left[\frac{r^4}{64} - \frac{r^5}{225\beta a}\right] + C_2\frac{r^2}{4} + C_3\log_e r + C_4$.

B.C's : $dw/dr = 0$ @ $r = 0$ (axi-symmetry) $\Rightarrow C_3 = 0$

$w = 0, \frac{dw}{dr} = 0$ at $r = a$: $\frac{dw}{dr} = 0 = \frac{p_0}{D}\cdot a^3\left[\frac{1}{16} - \frac{1}{45\beta}\right] + C_2\frac{a}{2}$

$\Rightarrow C_2 = -\frac{2p_0a^2}{D}\left[1/16 - 1/45\beta\right]$

$w = 0 = \frac{p_0a^4}{D}\left[\frac{1}{64} - \frac{1}{225\beta}\right] - \frac{p_0a^4}{2D}\left[\frac{1}{16} - \frac{1}{45\beta}\right] + C_4$

$-C_4 = \frac{p_0a^4}{D}\left[\frac{1}{64} - \frac{1}{32} - \frac{1}{225\beta} + \frac{5}{225\beta}\right] = \frac{p_0a^4}{D}\left[-\frac{1}{64} + \frac{1}{150\beta}\right]$

whence $\boxed{w = \frac{p_0}{D}\left\{\left[\frac{r^4}{64} - \frac{r^5}{225\beta a}\right] - \frac{a^2r^2}{2}\left[\frac{1}{16} - \frac{1}{45\beta}\right] + a^4\left[\frac{1}{64} - \frac{1}{150\beta}\right]\right\}}$

(c) if $w = 0$ in centre $\Rightarrow C_4 = 0 \Rightarrow \frac{1}{150\beta} = \frac{1}{64}$

$\Rightarrow \beta = 64/150$  $[\sim 0.43]$