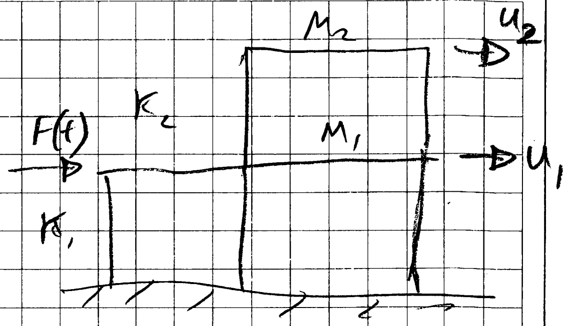
Company	Solution		DD5
Project	4D6 question	Reference No.	
Page	1 of 3	Date and place	

1) (a) $M_2 = 5 \times 200 = 1000\,kg$

$M_1 = 7.5 \times 200 = 1500\,kg$



$$K_2 = \frac{2 \times 12 \times 2500}{3^3} = \frac{20000}{9}\,kN/m$$

$$K_1 = \frac{3 \times 12 \times 5000}{3^3} = \frac{20000}{3}\,kN/m$$

Try mode shape $\bar{u} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ $\therefore$ $M_{eq} = 1 \times 1000 + \left(\frac{2}{3}\right)^2 \times 1500$
$= 1667\,kg$

$$K_{eq} = \frac{20000}{3} \times \left(\frac{2}{3}\right)^2 + \frac{20000}{9} \times \left(\frac{1}{3}\right)^2 = \frac{20000}{3^3}\left(4 + \frac{1}{3}\right) = \frac{13 \times 20000}{3^4}$$
$= 3210\,kN/m$

$$\therefore \quad T = 2\pi\sqrt{\frac{1667}{3210 \times 10^3}} = 0.143\,s$$

Try mode shape $\bar{u} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ $\therefore$ $M_{eq} = 1^2 \times 1000 + \left(\frac{1}{3}\right)^2 \times 1500 = 1167\,kg$

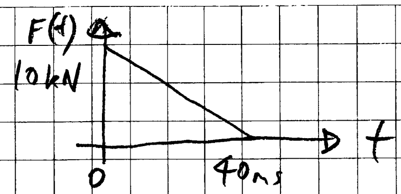
$$K_{eq} = \frac{20000}{3}\left(\frac{1}{3}\right)^2 + \frac{20000}{9} \times \left(\frac{2}{3}\right)^2 = \frac{20000}{3^3}\left(1 + \frac{4}{3}\right) = \frac{7 \times 20000}{3^4} = 1728\,kN/m$$

$$\therefore \quad T = 2\pi\sqrt{\frac{1167}{1728 \times 10^3}} = 0.163\,s$$

By Rayleigh's principle the mode shape $\bar{u} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ is more likely to be the fundamental mode as it has the longer period.

(b) $F_{eq} = \frac{1}{3} F(t)$

$= \frac{1}{3} \times 10\text{ kN}$



$t_d = 40\,ms = 40 \times 10^{-3}\,s$

$\therefore \frac{t_d}{T} = \frac{40 \times 10^{-3}}{0.163} = 0.245 \quad \therefore$ DAF ≈ 0.7 from data sheet

$\therefore \delta_{DYNAMIC} = \frac{\frac{1}{3} 10 \times 10^3}{1728 \times 10^3} \times 0.7 = 1.35 \times 10^{-3} = 1.35\,mm$

(c) $M_1 \ddot{u}_1 = K_2(u_2 - u_1) - K_1 u_1$

$M_2 \ddot{u}_2 = K_2(u_1 - u_2)$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$1000 \begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \frac{20 \times 10^6}{9} \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix} \sin \omega t$

$$\therefore \begin{bmatrix} \frac{20 \times 10^6}{9} \times 4 - 1000 \times \frac{3}{2} \omega^2 & -\frac{20 \times 10^6}{9} \\ -\frac{20 \times 10^6}{9} & \frac{20 \times 10^6}{9} - 1000 \omega^2 \end{bmatrix} \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\lambda = 1000 \omega^2 \Big/ \frac{20 \times 10^6}{9} = \frac{9 \omega^2}{20 \times 10^3}$

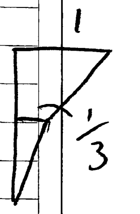
$\therefore \left(4 - \frac{3}{2}\lambda\right)(1 - \lambda) - 1 = 0$

$3 - \frac{11}{2}\lambda + \frac{3}{2}\lambda^2 = 0$

$(3\lambda - 9)\left(\lambda - \frac{2}{3}\right) = 0 \qquad \lambda = 3$ or $\frac{2}{3}$

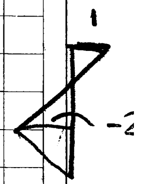
$$\begin{bmatrix} 4-\frac{3}{2}\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = 0$$

If $\lambda = \frac{2}{3}$ $\begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = 0 \quad \therefore \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$



Hence best mode shape from (a) is the correct fundamental mode

If $\lambda = 3$ $\begin{bmatrix} -\frac{1}{2} & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = 0 \quad \therefore \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



$$\frac{9\omega_2^2}{20\times10^3} = 3 \quad \therefore T_2 = \frac{2\pi}{\omega_2} = 0.077\,s$$

Mode 2 response

$F_{eq} = -2\times10 = -20\,kN$ $\qquad \frac{t_d}{T_2} = \frac{40\times10^{-3}}{0.077} = 0.519$

$\therefore DAF \simeq 1.23$

$$K_{eq} = \frac{20000}{3}\times(2)^2 + \frac{20000}{9}\times(3)^2 = 20000\left(\frac{4}{3}+1\right) = 46667\ kN/m$$

$$\delta_{dynamic} = \frac{-20\times10^3}{46667\times10^3}\times1.23 = -0.53\times10^{-3} = -0.53\,mm$$

Combine modes

$$SRSS \quad \delta = \sqrt{1.35^2 + (-0.53)^2} = 1.45\,mm$$

check: $M_{eq} = 1^2\times1000 + (-2)^2\times1500 = 7000\,kg$

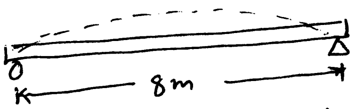
$$T_2 = 2\pi\sqrt{\frac{7000}{46667\times10^3}} = 0.077\,s$$

2 a) In a lumped mass system, it is assumed that all the masses are concentrated at floor levels while all the stiffness is concentrated in columns. These types of models are useful to analyse portal frame structures that undergo sway vibrations. [20%]

In a distributed mass system, the mass and stiffness are assumed to be distributed throughout the structure. Analysis is carried out using energy methods. These types of models are useful to analyse flexural vibration of a beam. [10%]

b) Consider the 1st mode of vibration $n = 1$. first.

Mode shape $= \bar{U}_1(x) = \sin\frac{\pi x}{8}$



Find equivalent stiffness, mass and force using data book.

$$K_{1eq} = \int_0^L EI\left(\frac{d^2\bar{U}_1}{dx^2}\right)^2 dx \qquad \frac{d^2\bar{U}_1(x)}{dx^2} = -\frac{\pi^2}{8^2}\sin\frac{\pi x}{8}.$$

$$\therefore K_{1eq} = \int_0^L EI\left(-\frac{\pi^2}{8^2}\sin\frac{\pi x}{8}\right)^2 dx = EI\frac{\pi^4}{8^4}\int_0^8 \sin^2\frac{\pi x}{8}\,dx.$$

$$= EI\frac{\pi^4}{8^4}\left[\int_0^8\left(\frac{1-\cos\frac{2\pi x}{8}}{2}\right)dx\right] = EI\frac{\pi^4}{8^4}\times\frac{8}{2}$$

Substituting $EI = 6\times10^5\ \text{N-m}^2$

$K_{1eq} = 57.075$ kN/m.

$$M_{1eq} = \int_0^L m\,\bar{U}_1^2\,dx = \int_0^L m\sin^2\frac{\pi x}{8}\,dx = \int_0^L m\left[\frac{1-\cos\frac{2\pi x}{8}}{2}\right]dx$$

$$= \frac{mL}{2} = 400\times\frac{8}{2} = 1600\ \text{kg}.$$

$$\therefore \omega_1 = \sqrt{\frac{K_{1eq}}{M_{1eq}}} = \sqrt{\frac{57.075\times10^3}{1600}} = 5.97\ \text{rad/s}\ \text{or}\ \underline{0.95\ \text{Hz}}.$$

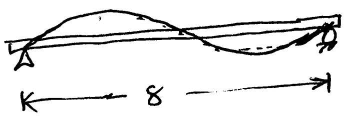
Equivalent force $F_{1eq} = F_0 \cdot \bar{U}_1 \big|_{x=\frac{3L}{4}}$ ← for point P

$= 15\times10^3 \times \sin\frac{\pi x}{8}\Big|_{x=6}$

$= 10.607$ kN.

Consider the 2nd mode of vibration of the beam.

mode shape $\bar{U}_2(x) = \sin\frac{2\pi x}{8} = \sin\frac{\pi x}{4}$



As before find equivalent stiffness, mass and force.

$$K_{2eq} = EI\int_0^L \left(\frac{d^2\bar{U}_2(x)}{dx^2}\right)^2 dx \qquad \frac{d^2U_2}{dx^2} = \frac{\pi^2}{4^2}\left(-\sin\frac{\pi x}{4}\right)$$

$$= EI\frac{\pi^4}{4^4}\int_0^8 \left(\sin^2 \pi x/4\right) dx = \frac{EI\pi^4}{4^4} \times \left[\int_0^8 \frac{\left(1-\cos\frac{\pi x}{2}\right)}{2}dx\right]$$

$$= \frac{EI\pi^4}{4^3} = \frac{6\times10^5\times\pi^4}{4^3} = 913.210 \text{ kN/m}.$$

$$M_{2eq} = \int_0^L m\bar{U}_2^2 dx = \int_0^L m \sin^2\frac{\pi x}{4} dx = \int_0^L m\frac{\left(1-\cos\frac{\pi x}{2}\right)}{2}dx$$

$$= \frac{mL}{2} = 400\times\frac{8}{2} = 1600 \text{ kg}$$

$$F_{2eq} = F_0\, U_2(x)\big|_{x=\frac{3L}{4}} = 15\times10^3\times\sin\frac{\pi x}{4}\Big|_{x=6} = -15 \text{ kN}.$$

for point P

∴ 2nd mode natural frequency will be

$$\omega_2 = \sqrt{\frac{K_{2eq}}{M_{2eq}}} = \sqrt{\frac{913.210\times10^3}{1600}} = 23.89 \text{ rad/s} \text{ or } \underline{\underline{3.802 \text{ Hz}}}$$

[40%]

2c) Find max deflections at point P in fundamental and 2nd modes.

Fundamental mode :- $F_{1eq} = 10.607$ kN.

$K_{1eq} = 57.075$ kN/m.

$$\therefore U_{1\,static} = \frac{F_{1eq}}{K_{1eq}} = \frac{10.607}{57.075} = 0.186 \text{ m}.$$

$t_d = 1s$ $\quad f_1 = 0.95\ Hz$ $\quad \therefore T_1 = \frac{1}{f_1} = 1.052$ sec.

$\therefore \frac{t_d}{T_1} = \frac{1}{1.052} = 0.95$.

From Data Sheets DAF $= 1.51$

$\therefore$ Max displacement in 1st mode $= U_1$ static $\times$ DAF

$= 0.186 \times 1.51 = 0.28086$.

$\simeq \underline{\underline{0.281\ m}}$. $\therefore$ At P $U_{1\,max} = 0.281 \times \sin \frac{6\pi}{8}$

$= 0.198 m$

Consider 2nd mode of vibration:

U_2 static $= \frac{F_2}{K_2 e_2} = \frac{-15}{913.210} = -0.0164$ m.

$t_d = 1s$ $\quad f_2 = 3.802\ Hz$ $\quad T_2 = \frac{1}{f_2} = 0.263$ sec.

$\therefore \frac{t_d}{T_2} = \frac{1}{0.263} = 3.802$

FRom data Sheets DA F $= 1.05$

$\therefore$ max displacement in 2nd mode $= 1.05 \times -0.0164 = -0.0172 m$

Super pose max displacements using criteria 1 & 2

i) Max displacements are added $= |U_1|_{max} + |U_2|_{max}$

$= 0.198 + 0.0172 = \underline{\underline{0.215}}$ m

ii) SRSS method $= \sqrt{U_1^2 + U_2^2} = \sqrt{0.198^2 + 0.0172^2}$

$= 0.\underline{199}$ m $\simeq 0.2 m$

method i) assumes that both peak displacements occur in both modes at the same time. This is unlikely. So SRSS method can be more accurate.

[40%]

3a) Loose sands will collapse and densify when subjected to shear loading from earthquakes. However when these soils are fully saturated the pore water with in the sand has no time to escape as the soil wishes to suffer volumetric contraction. Thus the tendency to suffer contractile volumetric strains is manifested as an increase in excess pore water pressure. This can lead to a reduction in the effective stress in soil. If the effective stress in the soil falls to near zero value, full liquefaction is said to have resulted. [20%]

3 b) The dimensions of the block foundation are 6m x 6m x 4m

$2l = 6$ m $\quad l/b = 1 \quad e/b = 1.3\dot{3}$. $\quad \nu = 0.3$ for sand.

$\therefore$ $2b = 6$ m $\quad l = 3 \quad b = 3$.

$e = 4$ m

Use Wolf's formulae from data sheets.

$$K_{hx} = \frac{Gb}{2-\nu}\left[6.8\left(\frac{l}{b}\right)^{0.65} + 2.4\right]\left[1+\left(0.33 + \frac{1.34}{1+l/b}\right)\left(\frac{e}{b}\right)^{0.8}\right]$$

$$= \frac{Gb}{2-\nu}\left[6.8+2.4\right]\left[1+\left(0.33+\frac{1.34}{2}\right)(1.3\dot{3})^{0.8}\right]$$

$$= \frac{3G}{1.7} \times 9.2 \times 2.26 \quad = 36.67\ G.$$

$= 36.67 \times 150 \times 10^6$ N/m

$\therefore$ Horizontal stiffness of sandy soil $= 5.5 \times 10^9$ N/m.

$$K_{zg} = \frac{Gb}{2-\nu}\left[3.1\left(\frac{l}{b}\right)^{0.75} + 1.6\right]\left[1+\left(0.25+0.25\frac{b}{l}\right)\left(\frac{e}{b}\right)^{0.8}\right]$$

$$= \frac{Gb}{2-\nu}\left[3.1+1.6\right]\left[1 + 0.5 \times (1.3\dot{3})^{0.8}\right]$$

$$= \frac{G \times 3}{1.7} \times 4.7 \times 1.629 = 13.51\ G.$$

$\therefore$ vertical stiffness of sandy soil $= 13.51 \times 150 \times 10^6$

$= 2.027 \times 10^9$ N/m.

$= 2.027 \times 10^9$ N/m

For rotational stiffness about y axis

$$K_{ry} = \frac{Gb^3}{1-\nu}\left[3.73\left(\frac{l}{b}\right)^{2.4}+0.27\right]\left[1+\frac{e}{b}+\frac{1.6}{0.35+(l/b)^4}\left(\frac{e}{b}\right)^2\right]$$

$$= \frac{Gb^3}{1-\nu}\left[4\right]\left[1+1.3\dot{3}+\frac{1.6}{1.35}\times(1.3\dot{3})^2\right]$$

$$= 17.76 \times \frac{3^3}{0.7} G = 685.08 \times G.$$

$\therefore$ Rotational stiffness of soil $= 685.08 \times 150 \times 10^6$

$= \underline{102.76} \times 10^9$ N/m.

[30%]

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3 c) Horizonatal natural frequency $= f_h = \frac{1}{2\pi}\sqrt{\frac{K_{hx}}{m}}$.

$$\therefore f_h = \frac{1}{2\pi}\sqrt{\frac{5.5 \times 10^9}{691\,200}}$$

$$= \underline{14.19} \text{ Hz}.$$

Vertical natural frequency $f_v = \frac{1}{2\pi}\sqrt{\frac{K_v}{m}}$

$$f_v = \frac{1}{2\pi} \times \sqrt{\frac{2.027 \times 10^9}{691\,200}}$$

$$= \underline{\underline{8.62}} \text{ Hz}.$$

Mass moment of inertia $= I = 2,995,200$ kg$\cdot$m^2.

$\therefore$ Rocking frequency $= \frac{1}{2\pi}\sqrt{\frac{K_{ry}}{I}}$

$$= \frac{1}{2\pi}\sqrt{\frac{102.76 \times 10^9}{2\,995\,200}} = \underline{\underline{29.48}} \text{ Hz}$$

Note: These are all quite high frequencies, so we should expect very little damage, if the soil is able to retain its stiffness during the earthquake.

[20%]

3d) The shear modulus of the soil has reduced to 3 MN/m²

$G = 3\ MN/m^2$

$\therefore$ Re calculate, K_{hx}, K_v & K_{ry}.

$K_{hx} = 36.67\ G$

$= 36.67 \times 3 \times 10^6 = 110.01 \times 10^6\ N/m$

$\therefore$ Natural frequency for horizontal vibrations (in the direction of shaking)

$$f_h = \frac{1}{2\pi}\sqrt{\frac{110.01 \times 10^6}{691\,200}} = 2.0\ Hz$$

$K_v = 13.51\ G = 13.51 \times 3 \times 10^6$

$= 40.53 \times 10^6\ N/m$

$\therefore$ Natural frequency for vertical vibrations f_v

$$f_v = \frac{1}{2\pi}\sqrt{\frac{40.53 \times 10^6}{691\,200}}$$

$= 1.22\ Hz$

$K_{ry} = 685.08\ G = 685.08 \times 3 \times 10^6$

$= 2055.2 \times 10^6\ N/m$.

$I = 2995\,200\ kg\text{-}m^2$.

$\therefore$ Natural frequency for rocking vibrations f_r

$$f_r = \frac{1}{2\pi}\sqrt{\frac{K_{ry}}{I}} = \frac{1}{2\pi} \times \sqrt{\frac{2055.2 \times 10^6}{2995\,200}} = 4.16\ Hz$$

Due to degradation in soil stiffness, all the natural frequencies have come down significantly. Since we expect earthquakes to have energy present in the range of 1-5 Hz, we should expect the block foundations to resonate.

[30%].

4D6 2005.

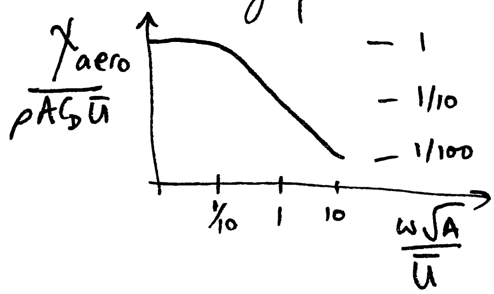
Q4. a) Mechanical admittance is a frequency-dependent transfer function that relates the power spectrum of an input force to the output displacement of a mechanical system.

e.g.

$|H(\omega)|^2$ = mechanical admittance.

Aerodynamic admittance is a similar frequency-dependent transfer function which relates input wind velocities to aerodynamic forces. It is a sort of "areal reduction factor" which takes account of the fact that eddies at different frequencies act over different length scales, and if that length scale is small c.f. the structure size, the effects of eddies at those frequencies will largely cancel each othe out.

e.g.



b) i) The modal mass is defined as $\displaystyle\int_{\text{structure}} m\,\phi^2\, dVol$,

where m is the mass density. In matrix form,

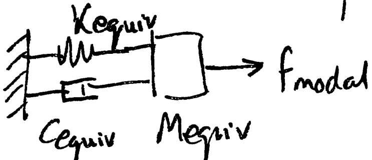
$$M_{modal} = \underline{\Phi}^T \underline{\underline{M}}\, \underline{\Phi} \quad \text{where } \underline{\Phi} \text{ is the mode shape.}$$

It takes account of the fact that, when oscillating in a particular mode, certain parts of the structure move more than other parts, and ~~???~~ so the inertia is "mobilised" more in some regions. Modal mass has a scale set by some convention – e.g. $\max(\underline{\Phi}) = 1$.

4D6, Q4, 2005.

b ii) Nonlinear geometric effects are important in suspension bridge structures. Significant ~~parts~~ proportions of the local tangent stiffness come from tension-stiffening and pendulum effects.

iii) By taking the mode shapes, frequencies and modal masses, and ~~idealise~~ rediscretising the bridge as a small number of independent linear s.d.o.f systems



The mode generalised force, $f_{modal}(t)$ can be calculated as the periodic walking forces modulated by $\Phi(x,t)$, the modal coordinate at the point where the person is walking.

Each s.d.o.f. oscillator can then be integrated forward in time simply, using say Runge-Kutta or Newmark β.

iv) Because phase synchronisation is not a random process ... etc.

c). Nat freq $n_s \sim \sqrt{\frac{K}{M}}$ so increasing M decreases natural frequency n_s. Lock-in when $St \simeq$ const $\simeq 0.2$ say

So, $U_{crit} \simeq \frac{n_v D}{0.2} \simeq \frac{n_s D}{0.2}$ since $n_v \simeq n_s$ at lock-in

$\therefore$ If n_s falls U_{crit} falls. Therefore it might not oscillate at such large amplitudes, but it might oscillate considerably more often (depending on local wind climate conditions), hence making fatigue problems worse.