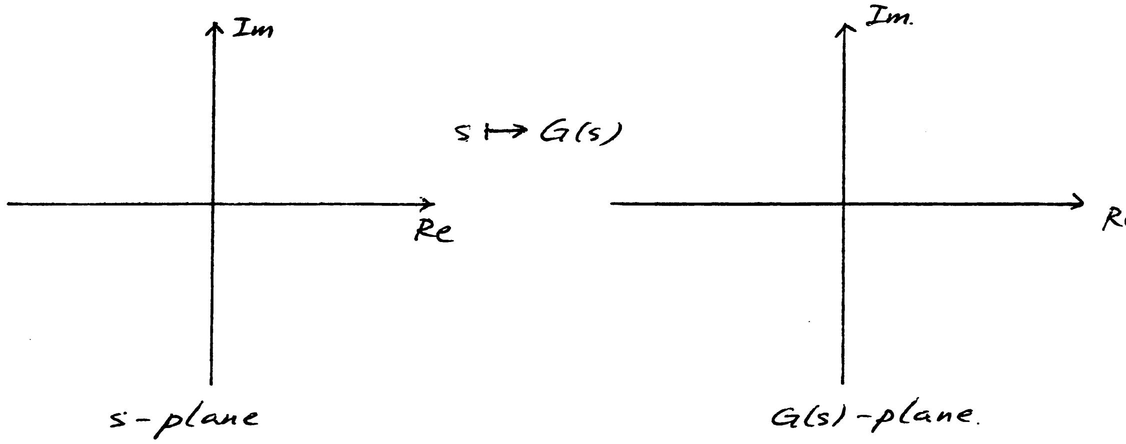
4F1 - 2005

1(a) [20%]

(i) The required mapping is $s \mapsto G(s)$.

Consider the two complex planes:



The root-locus diagram (for $k > 0$) consists of those points in the s-plane which map to the negative real axis in the G(s)-plane.

The Nyquist diagram is the image in the G(s)-plane of the imaginary axis in the s-plane.

The closed-loop poles are the values of s for which $1 + kG(s) = 0$. These are the values of s for which $G(s) = -1/k$, i.e., G(s) is real and negative.

[20%]

(ii). Let $G(s) = \alpha \dfrac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots}$ and assume $\alpha > 0$.

Then a point on the real axis is on the root locus if and only if there are an odd number of poles and zeros to the right of the point.

Proof:

If z_i (or p_i) is real, then $s-z_i$ (or $s-p_i$) is positive if s is real and greater than z_i (or p_i), and negative otherwise.

If there are an odd number of factors which are negative, then G(s) is negative.

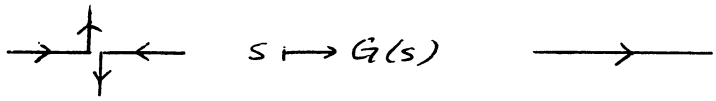
Any complex zeros or poles come in quadratic terms with complex roots, i.e., s^2+bs+c with $b^2 < 4c$, and such factors are always positive for real s.

[20%]

(iii) A map from s to $G(s)$ is conformal at all points where the derivative $\frac{d}{ds}G(s)$ both exists and satisfies $\frac{d}{ds}G(s) \neq 0$.

An important property of a conformal mapping is that, if two lines meet at an angle θ in the s-plane, then the mappings of these two lines also meet at an angle θ in the $G(s)$-plane.

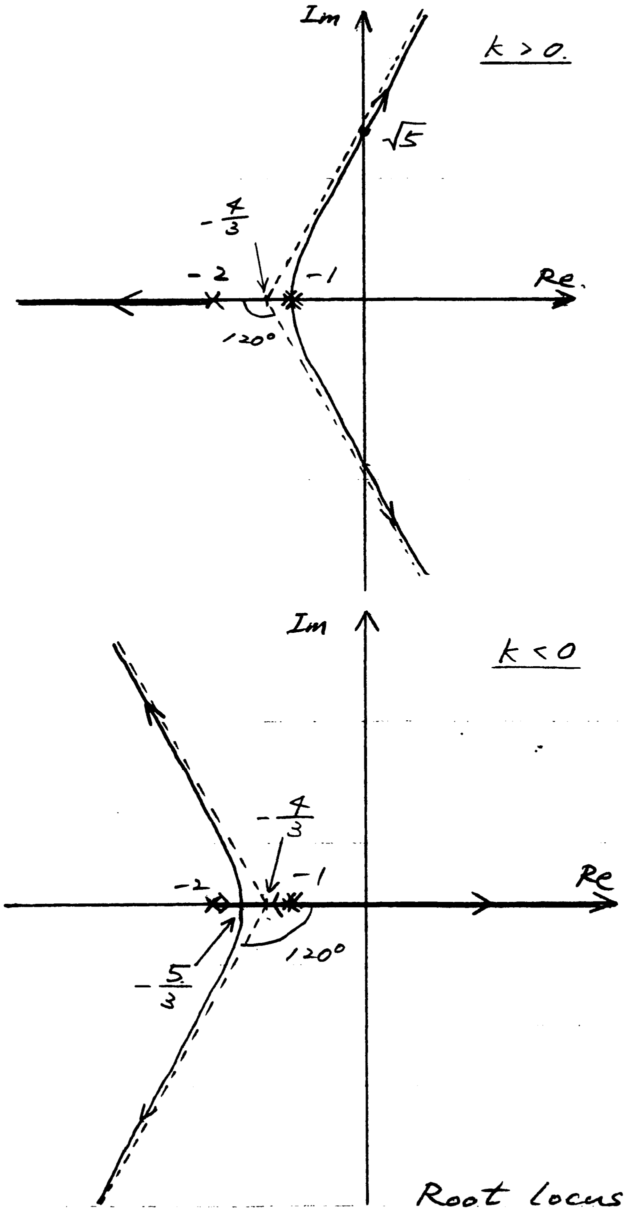
At breakaway points, the loci turn suddenly through a sharp angle as follows:



At such points, the mapping is not conformal.

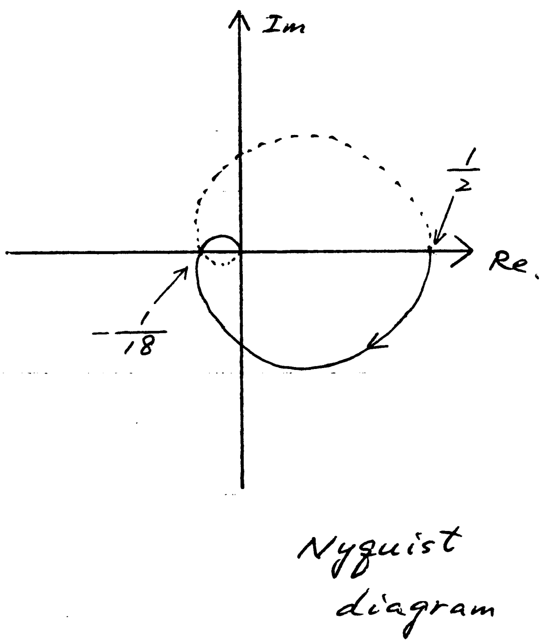
[40%]

(b)



Root locus diagram.

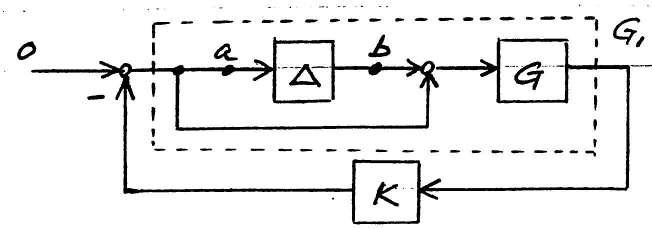
A sketch of the Bode diagram shows that the phase of $G(j\omega)$ decreases monotonically from 0° to −270°.



Nyquist diagram

2(a)

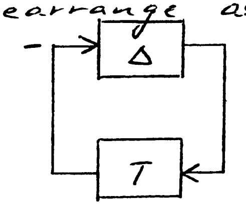
[20%]



$a = -KG(b+a)$

$\Rightarrow \quad a = \dfrac{-KG}{1+KG}\, b$

Rearrange as:



where $T = \dfrac{KG}{1+KG}$

Consider $T(s)$ to be stable and fixed. Let $h(\omega)$ be a positive and continuous function. The above feedback system is stable for all stable $\Delta(s)$ satisfying $|\Delta(j\omega)| \le h(\omega)$ (for all ω) if and only if

$$|T(j\omega)| < \frac{1}{h(\omega)} \qquad (*)$$

for all ω. (Small Gain Theorem)

Therefore (*) is the required condition.

(b)

[20%]

(i) The closed-loop poles for the nominal system are the roots of: $s^2 - 1 + k(s+1.25) = 0$.

For critical damping, this should equate to: $(s+\alpha)^2$.

This gives equations:

$$\left.\begin{aligned} k &= 2\alpha \\ 1.25k - 1 &= \alpha^2 \end{aligned}\right\} \Rightarrow \quad 5k - 4 = k^2$$

$$\Rightarrow \quad (k-1)(k-4) = 0$$

$k = 4$ and $\alpha = 2$ is the case with fastest possible decay rate.

(ii) $G_1(s) = \dfrac{s+1.25}{s^2-1}\Big(1 + \underbrace{\dfrac{(\delta_1 s + \delta_2)(s+2)}{s+1.25}}_{\Delta(s)}\Big)$

[15%]

$$T(s) = 4\frac{s+1.25}{s^2-1} \Big/ \Big(1 + 4\frac{s+1.25}{s^2-1}\Big) = \frac{4(s+1.25)}{s^2-1+4(s+1.25)}$$

$$= \frac{4(s+1.25)}{(s+2)^2}$$

(b) (cont'd)

Asymptote centre $= (-1-1-2)/3 = -\frac{4}{3}$

Breakaway points: $\frac{d}{ds} G(s) = 0$

$$0 = (s+1)^2 + 2(s+1)(s+2) = (s+1)(3s+5)$$
$$s = -1, \ -\frac{5}{3}$$

Real axis crossings of Nyquist diagram occur when $G(j\omega)$ is real.
The denominator of $G(j\omega)$ is $-j\omega^3 - 4\omega^2 + 5j\omega + 2$, and this is real when $\omega^3 - 5\omega = 0$, i.e., $\omega = 0, \pm\sqrt{5}$.
This gives $G(0) = \frac{1}{2}$ and $G(j\sqrt{5}) = -\frac{1}{18}$.
Note this also shows that imaginary axis crossings in the root locus diagram occur at $s = \pm j\sqrt{5}$ at a gain $k = 18$.

(ii) (cont'd)

The condition for robust stability then becomes:

$$\left| \frac{4(j\omega + 1.25)}{(j\omega + 2)^2} \right| < \frac{1}{h(\omega)}$$

for some $h(\omega)$ which satisfies $|\Delta(j\omega)| \le h(\omega)$ for all ω.

[25%]

(iii) For robust stability, it is sufficient that

$$\left| \frac{4(j\omega + 1.25)}{(j\omega + 2)^2} \right| \left| \frac{(\delta_1 j\omega + \delta_2)(j\omega + 2)}{j\omega + 1.25} \right| < 1$$

$$\Leftrightarrow \left| \frac{4(\delta_1 j\omega + \delta_2)}{j\omega + 2} \right| < 1$$

$$\Leftrightarrow |\delta_1| < 0.25, \; |\delta_2| < 0.5$$

where the last step follows by considering the cases $\omega = 0$ and $\omega = \infty$.

[20%]

(c) An integral relationship for $\log|S(j\omega)|$ holds in either of the following cases:

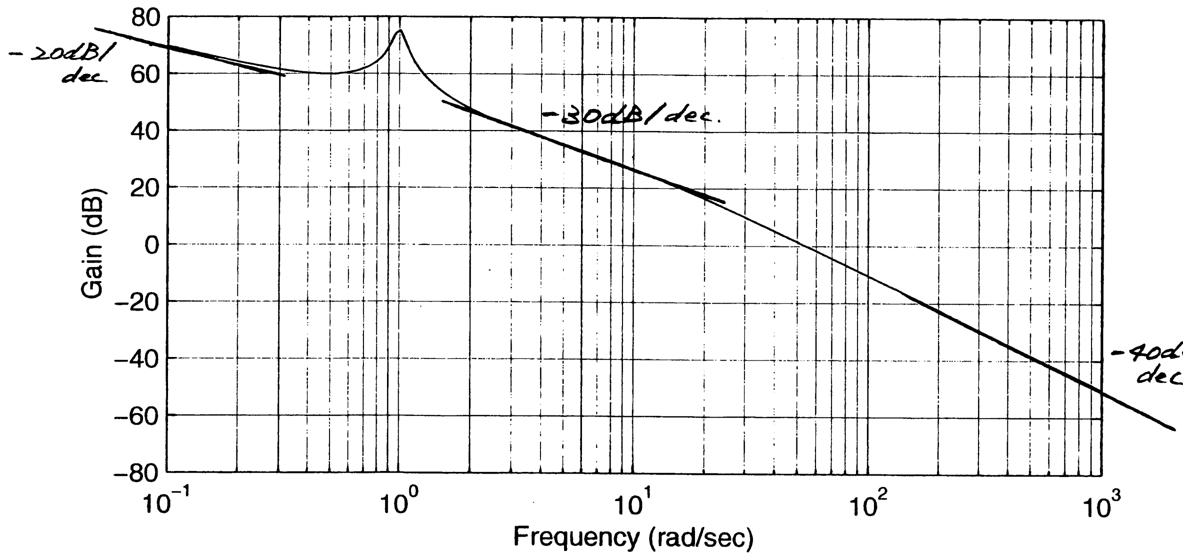
1. Either $G(s)$ or $K(s)$ has a RHP zero.
2. The return ratio $L(s)$ has at least second order roll-off at high frequency.

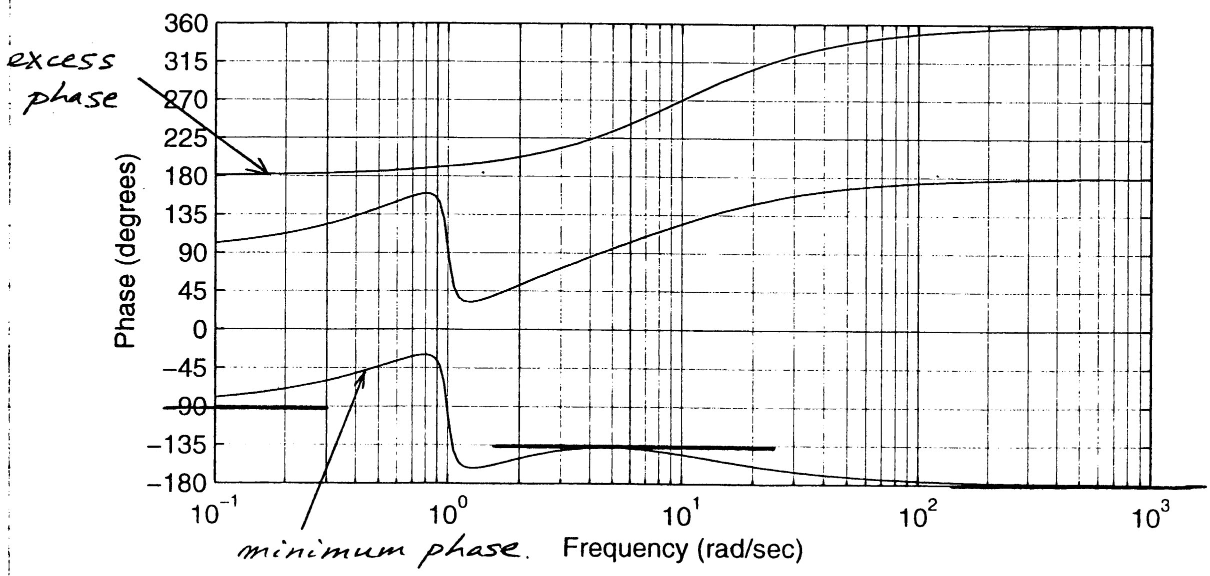
If the return ratio $L(s)$ is minimum phase and has first order high frequency roll-off, then it is possible for $|S(j\omega)|$ to be less than 1 for all frequencies, e.g.,

$$L = \frac{k}{s+1} \Rightarrow S = \frac{s+1}{s+1+k}.$$

3(a) **[20%]**

(i). Straight line approximations on the Bode diagram suggest an asymptotic phase of $-90°$ at low frequency and $-180°$ at high frequency.
The resonant peak in the magnitude plot is associated with a rapid decrease in the phase.
The true minimum phase is shown in the figure below.
An accurate sketch of this is not required.
The key feature to bring out is that the phase excess is $180°$ at low frequency and rises to $360°$ at high frequency.





[20%]

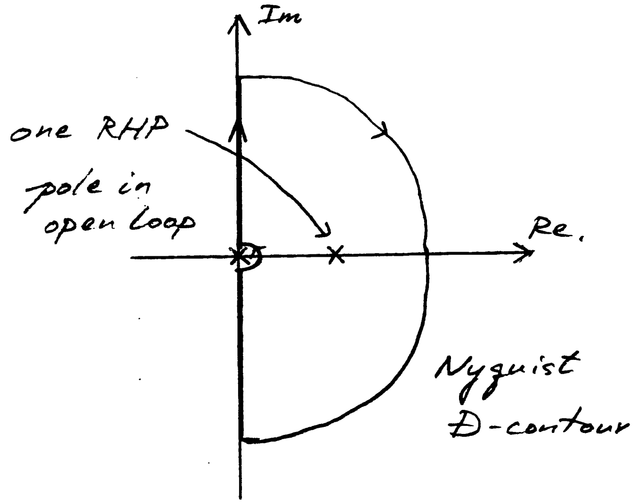
(ii) This suggests one <u>RHP</u> pole, with all-pass factor of the form $-\frac{\alpha+s}{\alpha-s}$, where the minus sign accounts for the 180° at d.c.
Phase excess is 270° around 10 rad/sec.
This suggests $\alpha = 10$.

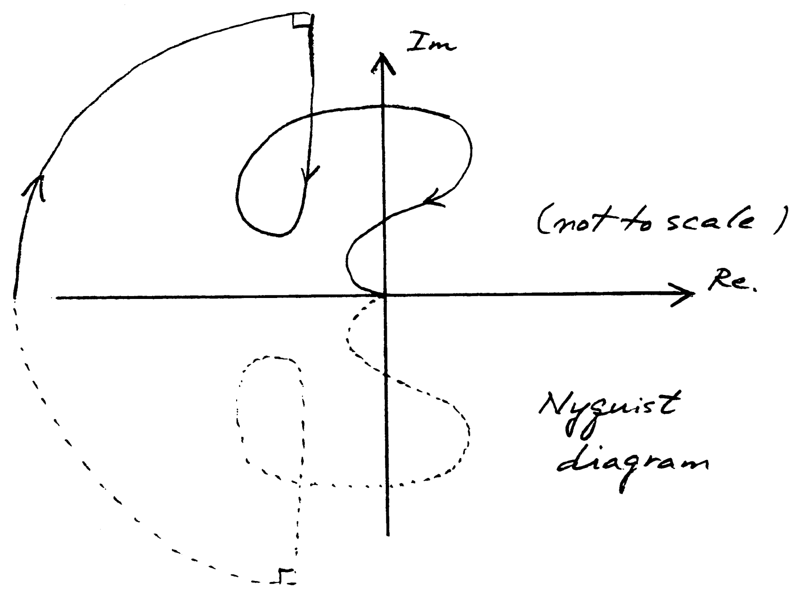
[10%]

(iii) This suggests that it would be difficult to achieve a gain crossover frequency of much less than 10 rad/sec with a conventional loop shape

[20%]

(b) At low frequency the roll-off rate is about -20dB/dec
This suggests one pole at the origin.





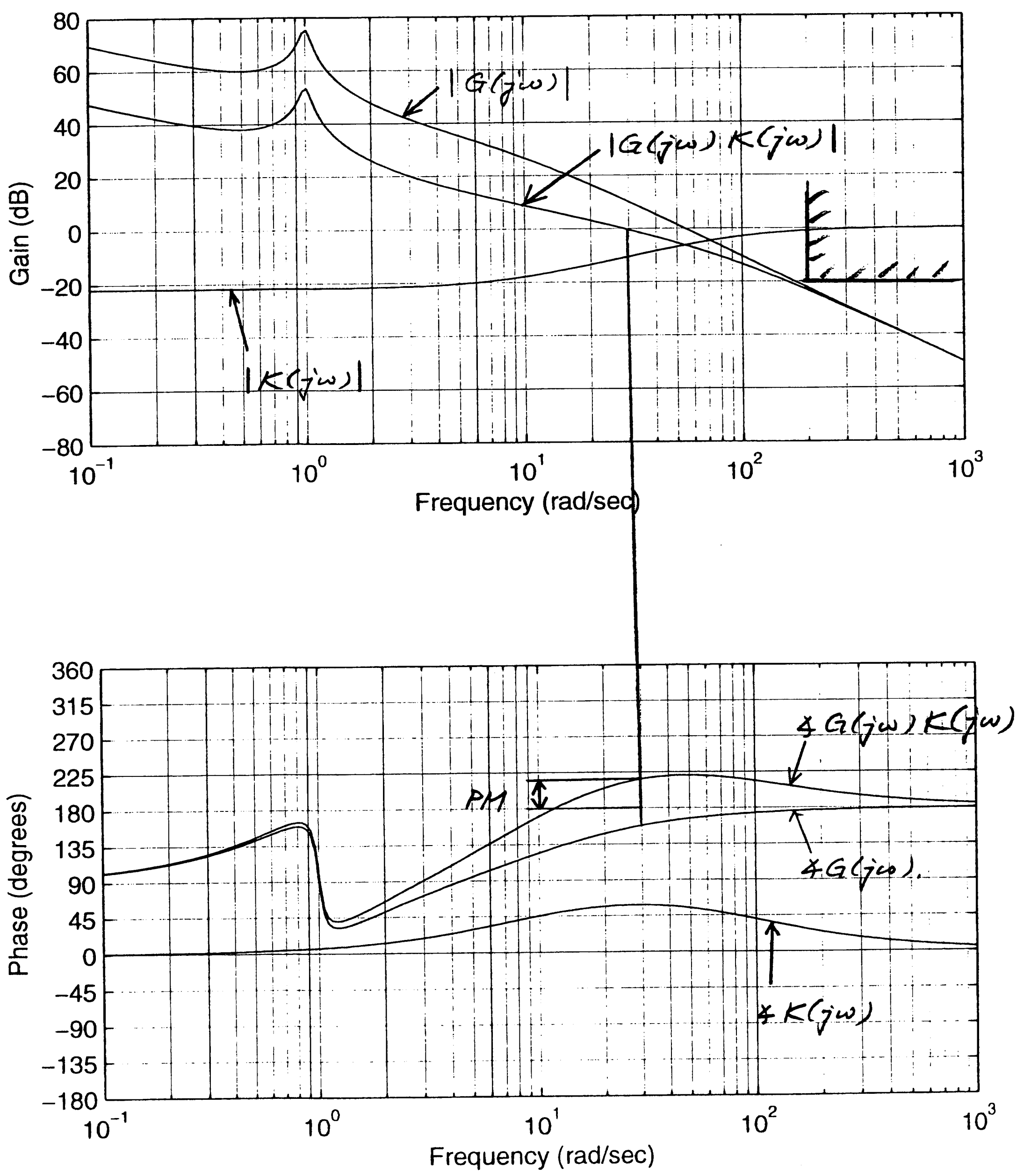
For $k > 0$, Nyquist diagram encircles $-1/k$ once clockwise.
" $k < 0$, " " " " zero times.

For closed-loop stability, need 1 anti-clockwise encirclement of $-1/k$. Hence
For $k > 0$, closed-loop has 2 poles in RHP,
" $k < 0$, " " " 1 pole " ".

[30%]

(c). Spec C. requires $|G(j\omega)K(j\omega)| \le 0.1$ approximately for $\omega \ge 200$ rad/sec.

This suggests that $|K(j\omega)| \simeq 1$ at high frequency. To achieve closed-loop stability and required phase margin, we need to increase the phase of $G(j\omega)K(j\omega)$ to above $210°$ somewhere in the range $10 \sim 100$ rad/sec. We also need to achieve a gain crossover frequency in this range. This can be achieved with a lead compensator, e.g., $K(s) = \frac{s+8}{s+100}$.



(c) (cont'd)

The compensated Nyquist diagram: