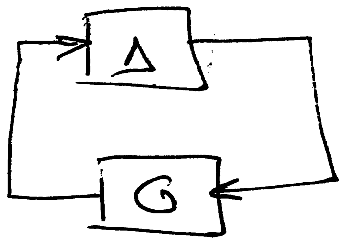
2005 IIB 4F2 Dr G Vinnicombe gv@eng.cam.ac.uk

1 (a)



the feedback interconnection is stable for all $\Delta \in H_\infty$, $\|\Delta\|_\infty < 1$ if and only if $G \in H_\infty$ and $\|G\|_\infty \leq 1$.

H_∞ is the space of transfer functions matrices satisfying $\bar{\sigma}(G(s)) < \infty$ $\forall_{s \in Re(s)>0}$. For $G \in H_\infty$, $\|G\|_\infty = \sup_{s \in Re(s)>0} \bar{\sigma}(G(s))$

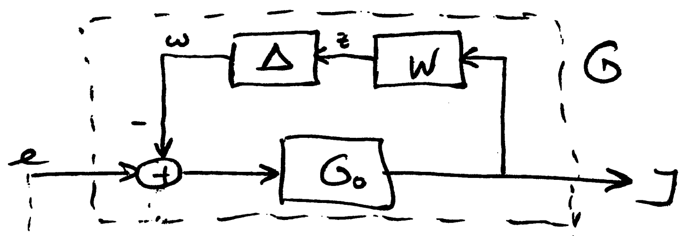
$$= \sup_{\omega} \bar{\sigma}(G(j\omega)).$$

(b) First note that $G_0(I+\Delta W G_0)^{-1} = (I+G_0\Delta W)^{-1} G_0$

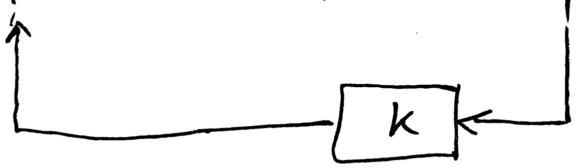
(proof: $G_0 = (I+G_0\Delta W)^{-1} G_0 (I+\Delta W G_0)$
$= (I+G_0\Delta W)^{-1}(I+G_0\Delta W) G_0 = G_0$)

thus, writing $y = Ge = (I+G_0\Delta W)^{-1} G_0 e$

or $y = G_0 e - G_0 \Delta W y$. In a block diagram:



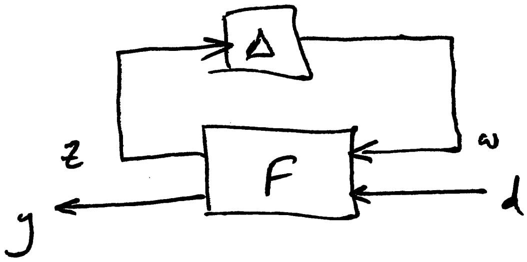
with a feedback controller k:



Now, $z = Wy$ and $y = G_0(ky - w)$ or $y = -(I-G_0k)^{-1} G_0 w$

$\Leftrightarrow$ $z = -W(I-G_0k)^{-1} G_0 w$. Using small gain theorem the system is stable for all Δ, $\|\Delta\|_\infty < \varepsilon$ iff $\|W(I-G_0k)^{-1}$ …

(c) we want to find F of the form



$$z = -W(I-G_0K)^{-1}G_0\, w + W(I-G_0K)^{-1}G_0K d$$

$$y = -(I-G_0K)^{-1}G_0\, w + (I-G_0K)^{-1} d$$

So,
$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} -W(I-G_0K)^{-1}G_0 & W(I-G_0K)^{-1}G_0K \\ -(I-G_0K)^{-1}G_0 & (I-G_0K)^{-1} \end{bmatrix} \begin{bmatrix} w \\ d \end{bmatrix}$$

the robust performance test is then

$$\mu \left\{ \begin{bmatrix} \varepsilon & \\ & \frac{1}{\alpha} \end{bmatrix} F \right\} \leq 1$$

with respect to the structure $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$.

(d) to find the closed loop poles:

$$\frac{10K}{s+2+\delta} + 1 = \frac{10K + s + 2 + \delta}{s+2+\delta} = \frac{s+2+10K+\delta}{s+2+\delta}$$ → Characteristic polynomial gives poles at $-2-\delta-10K$

For nominal stability ($\delta = 0$) need $2 + 10K > 0$ or $K > -\frac{1}{5}$

For robust stability need $\mathrm{Re}\{-2-\delta-10K\} < 0 \quad \forall |\delta| < 1$ or

$10K > -2 - \mathrm{Re}(\delta) \quad \forall |\delta| < 1 \iff 10K > -2+1 = -1 \iff K > -\frac{1}{10}$

("worst" perturbation when $\delta = -1$)

(e) $T_{d \to y} = \frac{1}{1+GK} = \frac{s+2+\delta}{s+2+10K+\delta}$

Using the final value theorem,

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \, T_{d \to y} \underbrace{\frac{1}{s}}_{\text{step input}} = \lim_{s \to 0} \frac{s+2+\delta}{s+2+10K+\delta}$$

$$= \frac{2+\delta}{2+10K(0)+\delta} = 0$$

$\Rightarrow$ Need $K(0) = \infty$

take $K(s) = \frac{1}{s}$. then $T_{d \to y} = \frac{s(s+2+\delta)}{s^2+(2+\delta)s+10}$

since $\mathrm{Re}(2+\delta) > 0$ the system is robust stable and satisfies the performance requirements.

2) a) $V(x,k) = \min_{u_k} \max_{w_k} \min_{u_{k+1}} \max_{w_{k+1}} \cdots$

$$\left(\sum_{l=0}^{h} c(x_l, w_l, u_l) + J_h(x_h) \right)$$

where $x_k = x$ etc

This is the optimal "cost to go" from the kth time step, where w, the disturbance, is trying to maximize the cost and u, the control, is trying to minimize it. u_k is chosen first (w_k is allowed to depend on it)

b) $V(x,2) = \min_u \max_w \; u^2 - w^2 + (x+w+u)^2$

$= \min_u \max_w \; 2u^2 - \cancel{w^2} + x^2 + \cancel{w^2} + \cancel{x^2} + 2xw + 2wu + 2ux$

$\underbrace{\qquad}_{2w(x+u)}$

$\Rightarrow w_2 = \mathrm{sgn}(x+u)$

want $\min_u \left(2u^2 + x^2 + 2|x+u| + 2ux \right)$

$\frac{d}{du} = 4u + 2x + 2\,\mathrm{sgn}(x+u) = 0$

$x+u$ +ve $\quad \cancel{2u^2+2u} + 2(u^2 + u + ux)$

$= 2(u + \frac{1+x}{2})^2$

$u_2 = -\frac{1}{2}\{x + \mathrm{sgn}(x+u_2)\}$ $\quad u = -\frac{1+x}{2}$

$x=0 \Rightarrow u=0 \Rightarrow 0$

$x = +ve \Rightarrow u$ -ve $\quad$ try $|u|<|x|$ $\quad u^2 + |x+u| + \cancel{2}ux$

$x=1$
$0 \Rightarrow 1$
$-1 \Rightarrow 1-1=0$

$- \; -x_2 \quad \Rightarrow V(x,1) = x^2$

$\Rightarrow$ Value function remains quadratic ~

$\Rightarrow$ $w_1 = \mathrm{Sgn}(x_1 + u_1)$

$u_1 = -x_1$ etc

$\Rightarrow$ optimal cost is 25

$u = -5, 0$

$w = 0, 0$!

3) a) $\sup_{\omega} \bar{\sigma}(G(j\omega))$

b) $\|y\|_2^2 < \gamma^2 \|u\|^2$

c) $\dot{V} = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA + C^T C & XB \\ B^T X & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$

$$\Rightarrow \max_u \dot{V} = x^T \big(\underbrace{A^T X + XA + C^T C + \frac{1}{\gamma^2} XBB^T X}_{=0}\big) x \Rightarrow \dot{V} \leq 0$$

$$\Rightarrow \|y\|_2^2 \leq \gamma^2 \|u\|^2 + \cancel{x_0^T X x_0}^{=0}$$

d) $-2x + 1 + \frac{1}{\gamma^2} x^2 = 0$

$\Rightarrow x > 0$ if $\gamma > 1$

$G(s) = \frac{1}{s+1} \Rightarrow \|G\|_\infty = 1$