4F3 Solutions 2005

1. (a) For the system $\dot{x} = f(x)$:

(i) S is an invariant set if $x(t) \in S \Rightarrow x(t+T) \in S$ for all $T > 0$.

(ii) x_e is an equilibrium point if $f(x_e) = 0$.

(iii) An equilibrium point x_e is stable if, for every $\varepsilon > 0$, there exists $\delta > 0$, such that $\|x(0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \varepsilon$ for all $t > 0$.

(iv) An equilibrium point x_e is globally asymptotically stable if:
- it is stable, and
- $x(t) \rightarrow x_e$ as $t \rightarrow \infty$, for every $x(0)$.

(b) At an equilibrium we have:

$$0 = a x_1 - x_1 x_3 + 2u \quad (1)$$
$$0 = -2x_2 + 2u \quad (2)$$
$$0 = b x_1 (x_1 - x_2) \quad (3)$$

$(2) \Rightarrow x_2 = u$. $\quad (3) \Rightarrow x_1 = 0$ or $x_1 = x_2$.

But $x_1 = 0 \Rightarrow u = 0$ (by (1)) which contradicts $u \neq 0$, which is given.

Hence $x_1 = x_2 = u$.

Then from (1) we have $0 = u(a - x_3 + 2)$

hence $x_3 = a + 2$.

So the (only) equilibrium is $\underline{\underline{(u, u, a+2)}}$.

[Note: This could be guessed from the form of V in part (c).]

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1 contd.

(c) $V = b(x_1 - x_2)^2 + b(x_2 - u)^2 + (x_3 - 2 - a)^2$

> 0 if $b > 0$ and $(x_1, x_2, x_3) \neq (u, u, 2+a)$.

$$\nabla V^T = \left[2b(x_1 - x_2),\ 2b(x_2 - u) - 2b(x_1 - x_2),\ 2(x_3 - 2 - a) \right]$$

$\therefore\ \dot{V} = \nabla V^T \underline{f}(\underline{x})$ [or by direct differentiation of V]

$$= 2b(x_1 - x_2)(ax_1 - x_1 x_3 + 2u) + (4bx_2 - 2bx_1 - 2bu)(-2x_2 + 2u) + 2(x_3 - 2 - a)bx_1(x_1 - x_2)$$

$$= 2b(x_1 - x_2)(ax_1 - x_1x_3 + 2u + x_1x_3 - 2x_1 - ax_1) + 4b(2x_2 - x_1 - u)(u - x_2)$$

$$= 4b(x_1 - x_2)(u - x_1) + 4b(x_2 - x_1 + x_2 - u)(u - x_2)$$

$$= 4b(x_1 - x_2)(u - x_1) - 4b(x_1 - x_2)(u - x_2) - 4b(u - x_2)^2$$

$$= 4b(x_1 - x_2)(u - x_1 - u + x_2) - 4b(u - x_2)^2$$

$$= -4b(x_1 - x_2)^2 - 4b(u - x_2)^2$$

< 0 if $x_1 \neq x_2$ or $x_2 \neq u$, since $b > 0$.

This establishes that the equilibrium is <u>stable</u>.

However, $\dot{V}(u, u, x_3) = 0$ for any x_3 (not just for $x_3 = 2 + a$).
So this does not prove asymptotic stability.

But suppose that $x_1 = x_2 = u$ and $x_3 \neq 2 + a$. Then, from equation (1), $\dot{x}_1 \neq 0$. Suppose this occurs at time t.
Then, since $\dot{x}_1(t) \neq 0$, we have $x_1(t + \tau) \neq u$ for some $\tau > 0$. Consequently $\dot{V}(t + \tau) < 0$.
Therefore, by La Salle's theorem, the equilibrium <u>is asymptotically stable</u>.

Since $\|x\| \to \infty \Rightarrow V \to \infty$, and $\dot{V} \leq 0$ for any $\underline{x}$, the whole state-space is a region of attraction for the equilibrium, so the asymptotic stability is <u>global</u>.

2 (a) <u>Circle criterion</u> for Fig. 1:

The system is globally asymptotically stable if the Nyquist locus $G(j\omega)$ encircles the circle with diameter $[-\frac{1}{\alpha}, -\frac{1}{\beta}]$ as many times counter-clockwise as $G(s)$ has right half-plane poles.

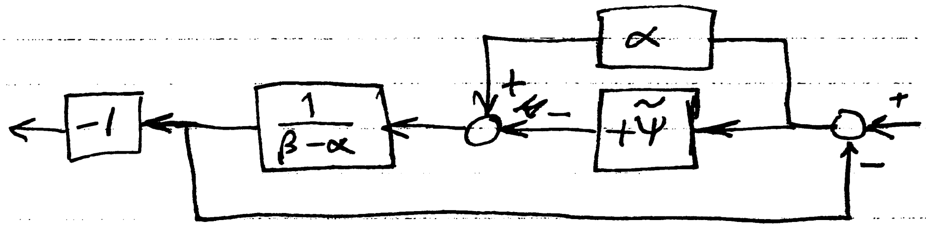
(b) $G(s)$ SPR and $\Psi(y) \geq 0 \Rightarrow$ system is g.a.s.

If $\Psi \geq 0$ ~~the~~ and $\cancel{\tilde{\Psi}}$ $\psi = \dfrac{\tilde{\psi} - \alpha}{\beta - \tilde{\Psi}}$

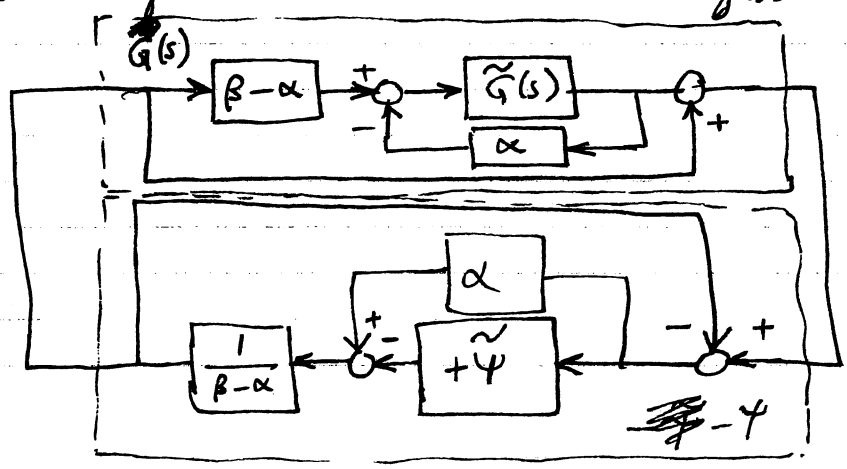
then $\alpha \leq \tilde{\Psi} \leq \beta$.

$$\text{But } \Psi = \frac{\tilde{\psi} - \alpha}{\beta - \tilde{\Psi}} = -\frac{\dfrac{\alpha - \tilde{\Psi}}{\beta - \alpha}}{1 + \dfrac{\alpha - \tilde{\Psi}}{\beta - \alpha}}$$

which can be obtained from Ψ by the following block-diagram:



Now the $\tilde{G} - \tilde{\Psi}'$ loop can be redrawn, with this realisation of $\tilde{\Psi}$ cancelled out in the forward path:



2 contd.

From the diagram,

$$G(s) = (\beta-\alpha)\frac{\tilde{G}(s)}{1+\alpha\tilde{G}(s)} + 1$$

$$= \frac{\beta\tilde{G}(s) + 1}{1+\alpha\tilde{G}(s)}$$

If $G(s)$ is strictly positive-real, what is the corresponding condition on $\tilde{G}(s)$?

$G(s)$ SPR $\Rightarrow$ $Re\{G(j\omega)\} \geq 0$

From the above we have

$$G(s)\left(1+\alpha\tilde{G}(s)\right) = \left[\beta\tilde{G}(s)+1\right]$$

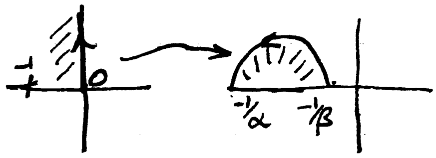
$$\Rightarrow \quad \tilde{G}(s)\left[\alpha G(s) - \beta\right] = 1 - G(s)$$

$$\Rightarrow \quad \tilde{G}(s) = \frac{G(s) - 1}{\beta - \alpha G(s)}$$

The imaginary axis is mapped by this transformation to a circle with real centre (given).

Clearly $G(s) = 0 \Rightarrow \tilde{G}(s) = -1/\beta$,

and $|G(s)| \to \infty \Rightarrow \tilde{G}(s) \to -1/\alpha$.



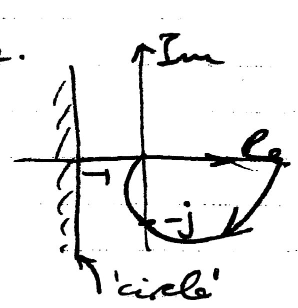
and $G(s) = -1 \Rightarrow$ ~~$\tilde{G}(s)$~~ $\frac{-2}{\beta+\alpha}$

Since the LHP corresponds to the interior of the circle (conformal mapping) and $G(j\omega)$ must not enter the LHP, $\tilde{G}(j\omega)$ must not enter the circle.

Also, $G(s)$ must be stable if it is to be SPR. So if $\tilde{G}(s)$ is unstable, the Nyquist locus $\tilde{G}(j\omega)$ must encircle $-1/\alpha$ as many times CCW as $\tilde{G}(s)$ has unstable poles (Nyquist theorem). But it cannot enter the circle, so it must encircle the circle this many times.

2 (c) If $\alpha = 0$ and $\beta = 1$, the circle criterion says that
contd.

$$Re\{G(j\omega)\} \geq -1 \text{ , since } G(s) \text{ is stable.}$$



We have $G(s) = \dfrac{2}{(s+1)^2}$

Either: Note that for $\omega = 1$

we have $G(j\omega) = G(j1) = \dfrac{2}{(j+1)^2} = \dfrac{2}{-1+2j+1} = -j$

so that $|G(j1)| = 1$ and $\angle G(j1) = -\pi/2$.

But $|G(j\omega)|$ is monotonically decreasing,

and $\angle G(j\omega)$ is monotonically decreasing from 0,

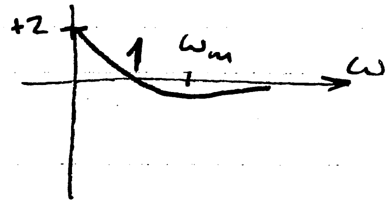
so $|G(j\omega)| < 1$ in the LHP.

Therefore $Re\{G(j\omega)\} \geq -1$ for all ω, and the circle criterion is satisfied. Hence g.a.s. inferred.

Or: $G(j\omega) = \dfrac{2}{(j\omega+1)^2} = \dfrac{2(-j\omega+1)^2}{(\omega^2+1)^2}$

$$= \frac{2(-\omega^2 - 2j\omega + 1)}{(\omega^2+1)^2}$$

$\therefore$ ~~$Im\{G(j\omega)\} = \frac{-4\omega}{(\omega^2+1)^2}$~~



$$\therefore Re\{G(j\omega)\} = \frac{2(1-\omega^2)}{(1+\omega^2)^2}$$

and this has a minimum at $\omega = \omega_m$, say.
Find this as follows:

$$\frac{d}{d\omega}\left[\frac{2(1-\omega^2)}{(1+\omega^2)^2}\right] = \frac{-4\omega(1+\omega^2)^2 - 2(1-\omega^2)2(1+\omega^2)2\omega}{(1+\omega^2)^4}$$

so $\frac{d}{d\omega}(\cdots) = 0$ when $4\omega(1+\omega^2)^2 + 8\omega(1-\omega^4) = 0$

ie at $\underbrace{\omega = 0}_{max}$ and $(1+2\omega^2+\omega^4) + 2(1-\omega^4) = 0$

2 contd.

or $3 + 2\omega^2 - \omega^4 = 0$

so $(\omega^2 - 3)(\omega^2 + 1) = 0$

hence $\underbrace{\omega^2 = 3}$ or $\underbrace{\omega^2 = -1}_{\text{not a real frequency}}$

hence $\omega_M = \sqrt{3}$.

Now $Re\left\{ G(j\sqrt{3}) \right\} = \dfrac{2(1-3)}{(1+3)^2} = \dfrac{-4}{16} > -1$

so the circle criterion is satisfied.

(d)

~~Circle criterion~~	Advantages	Disadvantages
Circle criterion:	Guarantees stability when condition satisfied.	Sufficient condition only. Stability not known if not satisfied.
Describing function	Can always apply it. Gives limit cycle prediction if limit cycle exists.	Approximate. Relies on low-pass assumption.

A.3 (a) $x_{k+1} = 0.2x_k + 0.5u_k + 0.8d_k$

$u_k = K(x_k - r_k)$

$\Rightarrow x_{k+1} = 0.2x_k + 0.5K(x_k - r_k) + 0.8d_k$

$= \cancel{0.7x_k - 0.5r_k + 0.8d_k}$

$= (0.2 + 0.5K)x_k - 0.5Kr_k + 0.8d_k$

~~Steady-state~~: $\cancel{0.3x}$

$= (0.2 - 0.1433)x_k + 0.1433r_k + 0.8d_k$

At steady-state:

$0.9433x_{ss} = +0.1433r_{ss} + 0.8d_{ss}$

$= +0.1433 \times 26 + 0.8 \times 18$

$= +3.7258 + 14.4$

$= \cancel{10.6742}\ 18.1258$

$\Rightarrow x_{ss} = \cancel{11.3158}\ 19.215\ °C$ ~~Not as required.~~ ~~Eric's sol has opposite sign of K~~ OK

(b) $u = u_\infty + K(x - r)$

At steady-state:

$x_{ss} = 0.2x_{ss} + 0.5[u_\infty + K(x_{ss} - r_{ss})] + 0.8d_{ss}$

$= [0.2 + 0.5K]x_{ss} + 0.5u_\infty - 0.5Kr_{ss} + 0.8d_{ss}$

$\Rightarrow (0.8 - 0.5K)x_{ss} = 0.5u_\infty - 0.5Kr_{ss} + 0.8d_{ss}$

But we want $x_{ss} = r_{ss}$

$\Rightarrow (0.8 - 0.5K + 0.5K)r_{ss} = 0.5u_\infty + 0.8d_{ss}$

$\Rightarrow \underline{\underline{u_\infty = 1.6(r_{ss} - d_{ss})}}$

(c) $0 \le u \le 10$, $20 \le x \le 30$.

Assuming u_∞ as in (b), at steady-state we have $x = r$ hence $u = u_\infty = 1.6\,(r_{ss} - d_{ss})$.

So $0 \le 1.6(r_{ss} - d_{ss}) \le 10$

$\Rightarrow \quad 0 \le r_{ss} - d_{ss} \le 6.25.$ ①

From model of pool: $0.8x_{ss} = 0.5u_\infty + 0.8d_{ss}$

$\Rightarrow \quad x_{ss} = 0.625u_\infty + d_{ss}$

$\therefore \; 20 \le 0.625u_\infty + d_{ss} \le 30$ ②

We also have $20 \le r_{ss} \le 30$.

$\therefore$ ① $\Rightarrow \; -20 \ge -d_{ss} \ge \cancel{36.25} \; -23.75$

$\Rightarrow \quad 20 \le d_{ss} \le 23.75$

(c) At steady-state: $0.8x_{ss} = 0.5u_{ss} + 0.8d_{ss}$

or $x_{ss} = 0.625u_{ss} + d_{ss}$.

or (as in (b)): $u_{ss} = 1.6\,(x_{ss} - d_{ss})$.

$\therefore \quad 0 \le u \le 10$

$\Rightarrow \quad 0 \le 1.6\,(x_{ss} - d_{ss}) \le 10$

$\Rightarrow \quad 0 \le x_{ss} - d_{ss} \le 6.25$

But $\quad 20 \le x_{ss} \le 30$

$x_{ss} = 20$: $\quad 0 \le 20 - d_{ss} \le 6.25 \Rightarrow \cancel{d_{ss}\;20 \ge d_{ss} \ge 13.75}$

$\Rightarrow \underbrace{-20 \le -d_{ss} \le -13.75}$

$13.75 \le d_{ss} \le 20$

$x_{ss} = 30$: $\quad 0 \le 30 - d_{ss} \le 6.25$

$\Rightarrow \underbrace{-30 \le -d_{ss} \le -23.75}$

$23.75 \le d_{ss} \le 30. \Rightarrow$

$\Downarrow$

$\underline{13.75 \le d_{ss} \le 30}$

$0 \le u \le 10, \qquad 20 \le x \le 30$

$$0 \le u \le 10 : \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \le \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

$$x_1 = 0.2\,x(k) + 0.5\,u(k) + 0.8\,d(k) \Rightarrow \cancel{\begin{matrix} 20 \le 0.2x(k) + 0.5u(k) + 0.8d \\ 30 \ge (\quad\quad) \end{matrix}}$$

$$\Rightarrow \cancel{\begin{bmatrix} 0.5 & 0 \\ -0.5 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \le} \Rightarrow$$

$$20 \le 0.2x(k) + 0.5\,u(k) + 0.8d(k)$$
$$30 \ge \text{———————}$$

$$\Rightarrow -0.5u(k) \le -20 + 0.2x(k) + 0.8d(k)$$

and $\quad 0.5u(k) \le 30 - 0.2x(k) - 0.8d(k)$

$$\Rightarrow \begin{bmatrix} -0.5 & 0 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \le \begin{bmatrix} -20 \\ 30 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} x(k) + \begin{bmatrix} 0.8 \\ -0.8 \end{bmatrix} d(k)$$

$$x_2 = 0.2x_1 + 0.5u_1 \cancel{(k+1)} + 0.8d(k) \quad [\hat{d}(k+1) = d(k)]$$
$$= 0.2\left[0.2x(k) + 0.5u_0 + 0.8d(k)\right] + 0.5u_1 + 0.8d(k)$$
$$= 0.04\,x(k) + 0.1u_0 + 0.5u_1 + 0.96\,d(k)$$

$$20 \le \text{———————}$$
$$30 \ge \text{———————}$$

$$\Rightarrow -0.1u_0 - 0.5u_1 \le -20 + 0.04\,x(k) + 0.96d(k)$$
$$0.1u_0 + 0.5u_1 \le 30 - 0.04\,x(k) - 0.96\,d(k)$$

$$\Rightarrow \begin{bmatrix} -0.1 & -0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \le \begin{bmatrix} -20 \\ 30 \end{bmatrix} + \begin{bmatrix} 0.04 \\ -0.04 \end{bmatrix} x(k) + \begin{bmatrix} 0.96 \\ -0.96 \end{bmatrix} d(k)$$

$$\therefore J = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -0.5 & 0 \\ 0.5 & 0 \\ -0.1 & -0.5 \\ 0.1 & 0.5 \end{bmatrix}, \quad c = \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \\ -20 \\ 30 \\ -20 \\ 30 \end{bmatrix}$$

$$W = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.2 \\ -0.2 \\ 0.04 \\ -0.04 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.8 \\ -0.8 \\ 0.96 \\ -0.96 \end{bmatrix}$$

✓

4. (a)

Advantages	Disadvantages
Handles constraints	Computational complexity
Easy to understand	Relies on good model
Multivariable easy	
Handles dead-time etc.	

Expect illustrations based on paper-industry case-study.

(b) (i) $$V^*(Ax + Bu_0^*(x)) = V(Ax + Bu_0^*(x), u_0^*)$$

Notation?

$$V^*(Ax + Bu_0^*(x)) \leq V(Ax + Bu_0^*(x), u_0^*(x))$$

$$V^*(x) = V(x, u_0^*(x))$$

4(b) (i)

$$V(Ax+Bu_0^*(x),0) = (Ax+Bu_0^*(x))^T Q (Ax+Bu_0^*(x))$$
$$\cancel{+ u_0^*(x)^T R u_0^*(x)}$$
$$+ (A(Ax+Bu_0^*(x)))^T P$$
$$(A(Ax+Bu_0^*(x)))$$

$$= w_0^T Q w_0 + w_1^T P w_1$$

$$V^*(x) = V(x, u_0^*(x)) = x^T Q x + u_0^*(x)^T R u_0^*(x) +$$
$$+ x_1^T P x_1$$

where $x_1 = Ax + Bu_0^*(x) = w_0$

$$\therefore V^*(x) = x^T Q x + u_0^*(x)^T R u_0^*(x) + w_0^T P w_0$$
$$\cancel{+ (Ax + Bu_0^*}$$

$$\therefore V(Ax+Bu_0^*(x),0) - V^*(x) =$$
$$w_0^T Q w_0 + w_1^T P w_1 - x^T Q x - u_0^*(x)^T R u_0^*(x) - w_0^T P w_0$$
$$= w_0^T (Q-P) w_0 + w_1^T P w_1 - x^T Q x - u_0^*(x)^T R u_0^*(x)$$
$$= - w_0^T A^T P A w_0 + w_1^T P w_1 - x^T Q x - u_0^*(x)^T R u_0^*(x)$$
$$= - \cancel{w_1^T P w_1} + \cancel{w_1^T P w_1} - x^T Q x - u_0^*(x)^T R u_0^*(x)$$
< 0 if $x \neq 0$, since $Q > 0$ and $R > 0$.

By definition, $V^*(Ax+Bu_0^*(x)) \leq V(Ax+Bu_0^*(x),0)$

$\Rightarrow$ proved.

4(b) (ii)

$V^*(x)$ needs to be continuous.

$V^*(Ax+Bu_0^*(x)) \le V(Ax+Bu_0^*(x), 0)$

$V^*(x) > 0$ for $x \neq 0$

$V^*(0) = 0$