4F6.

1) (a) + (b) – bookwork – details filled in in final solution

c) 64 coins in box

One coin double headed – "event" ($H2$)

Rest are ordinary " $H1$

Clearly $p(H2|C) = \frac{1}{64}$

"C" – context of the problem.

Selected coin now flipped and result R1 is heads,

$$p(H2|C) + p(H1|C) = 1$$

$$\therefore \quad p(H1|C) = 1 - \frac{1}{64} = \frac{63}{64}.$$

Also, $p(R1|H2,C) = 1$ only one possibility

n

2/

1 cont.)

Also, if coin is fair, $p(R1 \mid H1, C) = \frac{1}{2}$.

$$\therefore \frac{p(H2 \mid R1, C)}{p(H1 \mid R1, C)} = \frac{p(H2 \mid C)}{p(H1 \mid C)} \times \frac{p(R1 \mid H2, C)}{p(R1 \mid H1, C)}$$

$$= \frac{1/64 \times 1}{63/64 \times 1/2 / 2} = \frac{4}{63}.$$

Also, $p(H2 \mid R1, C) + p(H1 \mid R1, C) = 1$

$$\therefore p(H2 \mid R1, C) = \frac{1}{1 + 63/2} = \frac{2}{65}.$$

Next result R2 is also Heads,

$\Downarrow$ need to assume $p(R1 \mid R2, H1, C) = p(R1 \mid H1, C)$ (independence)

$$p(H2 \mid R1, R2, C) = \frac{4}{69}$$

2) a) – bookwork.

b) The likelihood is

$$p(z|y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(z-y)^2}{2\sigma^2}\right)$$

$$\therefore p(z) = \int_{-\infty}^{\infty} p(z|y)\, p(y)\, dy$$

$$= \int \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(z-y)^2}{2\sigma^2}\right) \left(.5\,\delta(y-1) + .5\,\delta(y+1)\right) dy$$

$$= \frac{1}{2\sqrt{2\pi}\,\sigma}\left[\exp\left(-\frac{(z-1)^2}{2\sigma^2}\right) + \exp\left(-\frac{(z+1)^2}{2\sigma^2}\right)\right]$$

The posterior is given by

$$p(y|z) = \frac{p(z|y)\,p(y)}{p(z)} = \frac{\exp\left(-\frac{(z-y)^2}{2\sigma^2}\right)\left(\delta(y-1) + \delta(y+1)\right)}{\exp\left(-\frac{(z-1)^2}{2\sigma^2}\right) + \exp\left(-\frac{(z+1)^2}{2\sigma^2}\right)}$$

2 cont.)

c) To maximize $p(y|z)$ we have to choose $\hat{y} = \pm 1$ ~~otherwise~~ the density is zero.

Since $|z-y|$ maximizes the density, we choose $\hat{y}$ to be the closest value to z.

$\therefore \quad \hat{y}_{MAP} = \mathrm{sgn}(z).$

d) To find the mean square estimate we need

$$\hat{y}_{ms} = \int_{-\infty}^{\infty} y \, p(y|z) \, dy.$$

$$= \int y \frac{e^{-\frac{(z-y)^2}{2\sigma^2}} \left(\delta(y-1) + \delta(y+1)\right)}{e^{-(z-1)^2/2\sigma^2} + e^{-(z+1)^2/2\sigma^2}} \, dy.$$

$$= \frac{e^{z/\sigma^2} - e^{-z/\sigma^2}}{e^{z/\sigma^2} + e^{-z/\sigma^2}} = \tan\left(\frac{z}{\sigma^2}\right)$$

3) a) and b) bookwork.

c)
$$F_{ij} = E\left(\frac{\partial \ln p(r|\theta)}{\partial \theta_i}\right)\left(\frac{\partial}{\partial \theta_j}\ln p(r|\theta)\right)$$

$$\ln p(r|\theta) = -\frac{N}{2}\ln(2\pi\theta_2) - \frac{1}{2\theta_2}\sum_{l=0}^{N-1}(r(l)-\theta_1)^2$$

$$\therefore \frac{\partial}{\partial \theta_1}\ln p(r|\theta) = \frac{1}{\theta_2}\sum_{l=0}^{N-1}(r(l)-\theta_1)$$

$$\frac{\partial}{\partial \theta_2}\ln p(r|\theta) = -\frac{N}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{l=0}^{N-1}(r(l)-\theta_1)^2$$

3 c conr)

Second partial derivatives are:

$$\frac{\partial^2}{\partial \theta_1^2} \ln p(r|\theta) = -\frac{N}{\theta_2}.$$

$$\frac{\partial^2}{\partial \theta_1 \partial \theta_2} \ln p(r|\theta) = \left(-\frac{1}{\theta_2^2}\right) \sum_{l=0}^{N-1} (r(l) - \theta_1) = \frac{\partial^2}{\partial \theta_2 \partial \theta_1} \log p(r|\theta)$$

$$\frac{\partial^2}{\partial \theta_2^2} \ln p(r|\theta) = \frac{N}{2\theta_2^2} - \frac{1}{\theta_2^3} \sum (r(l) - \theta_1)^2$$

Taking expectations;

$$\therefore \quad F = \begin{pmatrix} \frac{N}{\theta_2} & 0 \\ 0 & \frac{N}{2\theta_2^2} \end{pmatrix}$$

$\therefore$ from the C-R bound,

$$E(\hat{\theta}_1 - \theta_1)^2 \geq \frac{\theta_2}{N}, \quad E(\hat{\theta}_2 - \theta_2)^2 \geq \frac{2\theta_2^2}{N}$$

4) (a) – bookwork.

b) Likelihood ratio is

$$\lambda(r) = \frac{p_1(r_1, r_2 \cdots r_m)}{p_0(r_1, r_2, \cdots r_m)} = \prod_i \frac{p_1(r_i)}{p_0(r_i)} = \prod_i \lambda_i(r_i).$$

$$p_1(r_i) = F \exp\left(-\frac{1}{N_0}\int_0^T (r_i(t) - s_i(t))^2 \, dt\right)$$

$$p_0(r_i) = F \exp\left(-\frac{1}{N_0}\int_0^T r_i(t)^2 \, dt\right)$$

$$\therefore \lambda_i(r_i) = \exp\left(-\frac{1}{N_0}\int_0^T s_i^2(t)\, dt\right) \exp\left(\frac{2}{N_0}\int_0^T r_i(t)\, s_i(t)\, dt\right)$$

$$\therefore \lambda(r) = \prod_i^M \exp\left(-\frac{E_i}{N_0}\right) \prod_i^M \exp\left(\frac{2}{N_0}\int_0^T r_i(t)\, s_i(t)\, dt\right)$$

where $E_i = \int_0^T s_i^2(t)\, dt$ signal energy.

4 cont.)

Taking logs.

$$-\frac{1}{N_0}\sum_i E_i + \frac{2}{N_0}\sum \int_0^T r_i(t)\, s_i(t)\, dt \geq \ln \lambda_0 .$$

$$\therefore \sum_i \int_0^T r_i(t)\, s_i(t)\, dt \geq \frac{1}{2} N_0 \ln \lambda_0 + \frac{1}{2}\sum_i E_i$$

E_i – energy (signal) in i^{th} pulse

λ_0 – threshold.