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1.

a). i.c. $u(x,0) = f(x)$

b.c. $u(0,t) = g(t)$

no far field b.c. is required as the wave is marching towards it.

b). using Tylor expansion at point (i,n)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = \frac{-u_i^n + u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2}{\Delta t} + c\,\frac{u_i^n - u_i^n + \frac{\partial u}{\partial x}\Delta x - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2}{\Delta x}$$

$$\left(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0\right)$$

$$= \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \frac{1}{2}(c\cdot\Delta x)\left(1 - \frac{c\Delta t}{\Delta x}\right)\frac{\partial^2 u}{\partial x^2} = 0 \quad ; \quad \text{let } k = \frac{1}{2}c\Delta x\left(1 - \frac{c\Delta t}{\Delta x}\right)$$

$$= \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = 0$$

c). The PDE transforms from hyperbolic type to parabolic type. $(\Delta \equiv 0)$

The original PDE has a solution which is constant along its characteristics $x = ct$. The new PDE ~~retained or~~ retains the characteristics but dissipation now is dominant along the characteristics: the solution decays exponentially along it. The parameter k controls the rate of decay (dissipation)

The numerical solution is exact when $\frac{c\Delta t}{\Delta x} = 1$, as all higher order ~~the~~ terms vanish and the discretised finite difference equation is exactly the original PDE.

d). for $k \geq 0$, the system is well-posed ~~because a~~ because a solution is unique and stable for given initial and boundary conditions.

for $k < 0$, $u(x,t)$ will grow exponentially along $x = ct$, thus for large x or t, the solution do not exist and is not stable.

2. $\because$ b.c. independent of θ. u must be axisymmetric, thus $u_\theta = u_{\theta\theta} \equiv 0$.

a). integrating along r twice and apply b.c.s

$$u(r,\theta) = \frac{3}{\ln 2} \cdot \ln r + 5$$

b). u is independent of θ and a monotonely increasing function of r, it has minimum $u(1)=5$ on $r=1$ and maximum $u(2)=8$ on $r=2$, for $r>1$, $u(r)>5$ and for $r<2$, $u(r)<8$. thus inside the domain $1<r<2$. $u(r)$ is smaller than its maximum on $r=2$ and larger than its minimum on $r=1$

c). Let both u_1 & u_2 satisfy $\nabla^2 u = 0$ and the same b.c. $u_\Gamma = f$. the difference of them $u' = u_1 - u_2$ must also satisfy $\nabla^2 u' = 0$ and homogeneous b.c. $u'_\Gamma = 0$. According to the maximum principle, inside the domain surrounded by Γ, u' cannot be larger than the maximum value on Γ nor smaller than the minimum value on Γ, which is zero, thus $u' \equiv 0$. which leads to $u_1 \equiv u_2$.
$\therefore$ solution is unique.

for stability, let u_1 be solution of $\nabla^2 u_1 = 0$ and $u_{1\Gamma} = f_1$, u_2 for $\nabla^2 u_2 = 0$ and $u_{2\Gamma} = f_2$. the difference $u_1 - u_2$, should satisfy: $\nabla^2(u_1 - u_2) = 0$ and $(u_1 - u_2)_\Gamma = f_1 - f_2$. According to maximum principle. for any given small number ε, if we have $|f_1 - f_2| \le \varepsilon$. $|(u_1 - u_2)_\Gamma| = |f_1 - f_2| \le \varepsilon$ and $|u_1 - u_2| < |(u_1 - u_2)_\Gamma| \le \varepsilon$. thus u is stable regarding to perturbations to b.c. $u_\Gamma = f$.

d). because $\nabla^2 u = 0$ there is no source in the domain thus $\oint \frac{\partial u}{\partial n} ds$ should be zero, i.e. $\int_{\Gamma_1} r_1 c_1 \, ds = \int_{\Gamma_2} r_2 c_2 \, ds$, $\therefore$ ~~$c_1 \cdot 1 = c_2 \cdot 2$~~ $2\pi r_1 \cdot c_1 = 2\pi r_2 \cdot c_2$,

$c_1 = 2 c_2$.

Also notice for Neumann problem the constant in the solution cannot be fixed, thus the solution is unique only in the sense that a constant difference is allowed for.

3 (a)(i) $\underline{u} \times (\nabla \times \underline{u})\big|_i = \varepsilon_{ijk} u_j \left(\varepsilon_{klm} \dfrac{\partial u_m}{\partial x_l} \right)$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \frac{\partial u_m}{\partial x_l}$$

$$= u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j}$$

$$= \frac{1}{2} \frac{\partial}{\partial x_i}(u_j u_j) - u_j \frac{\partial u_i}{\partial x_j}$$

$$= \left[\frac{1}{2} \nabla (\underline{u}.\underline{u}) - (\underline{u}.\nabla)\underline{u} \right]_i$$

(ii) Now $\rho \left(\dfrac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u} \right) = -\rho \nabla \psi - \nabla p$, $\nabla . \underline{u} = 0$

So using previous result,

$$\frac{\partial \underline{u}}{\partial t} + \frac{1}{2}\nabla(\underline{u}.\underline{u}) - \underline{u} \times (\nabla \times \underline{u}) = -\nabla \psi - \frac{1}{\rho} \nabla p$$

so with $\underline{w} = \nabla \times \underline{u}$,

$$\frac{\partial \underline{u}}{\partial t} - \underline{u} \times \underline{w} = -\nabla \left(\frac{p}{\rho} + \psi + \frac{1}{2} \underline{u}.\underline{u} \right) = -\nabla H \text{ sa}$$

Take curl, and write down ith component:

$$\frac{\partial w_i}{\partial t} - \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{klm} u_l w_m \right) = \underbrace{- \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial H}{\partial x_k} \right)}_{\substack{=0 \text{ because} \\ \text{symmetric/antisymmetric in } j,k}}$$

$$\therefore \frac{\partial w_i}{\partial t} - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \left(\frac{\partial u_l}{\partial x_j} w_m + u_l \frac{\partial w_m}{\partial x_j} \right) = 0$$

$$\therefore \frac{\partial w_i}{\partial t} = w_j \frac{\partial u_i}{\partial x_j} - w_i \underbrace{\frac{\partial u_j}{\partial x_j}}_{\nabla . \underline{u} = 0} + u_i \underbrace{\frac{\partial w_j}{\partial x_j}}_{\nabla . \underline{w} = \nabla . (\nabla \times \underline{u}) = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk} \frac{\partial u_j}{\partial x_k}\right) = 0} - u_j \frac{\partial w_i}{\partial x_j}$$

$$\therefore \left[\frac{\partial \underline{w}}{\partial t} + (\underline{u}.\nabla)\underline{w} - (\underline{w}.\nabla)\underline{u} \right]_i = 0$$

3 cont. (b)(i) In suffix notation, Stokes's theorem is

$$\iint \varepsilon_{ijk} \frac{\partial f_k}{\partial x_j} n_i \, dA = \oint f_k \, dl_k$$

let $f_k = \phi a_k$ where ϕ is a scalar field and $\underline{a}$ is a constant vector.

Then $$\varepsilon_{ijk} \iint \frac{\partial}{\partial x_j}(\phi a_k) n_i \, dA = \oint a_k \phi \, dl_k$$

$$\therefore \quad a_k \left\{ \varepsilon_{ijk} \iint \frac{\partial \phi}{\partial x_j} n_i \, dA - \oint \phi \, dl_k \right\} = 0$$

because $\underline{a}$ is constant. But for given ϕ, this is true for any vector $\underline{a}$. So could consider $\underline{a} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

to deduce that each separate component of $\{ \cdots \} = 0$

ie $$\iint \varepsilon_{ijk} \frac{\partial \phi}{\partial x_j} n_i \, dA = \oint \phi \, dl_k$$

By a similar argument ϕ could be replaced by any expression in suffix notation.

(ii) For plane regions, normal $\underline{n}$ is constant and can be pulled outside integral

$$\therefore \quad \varepsilon_{ijk} n_i \iint \frac{\partial \phi}{\partial x_j} dA = \oint \phi \, dl_k$$

Multiply by $\varepsilon_{kpq} n_p$, or in vector terms take $\underline{n} \times \cdots$

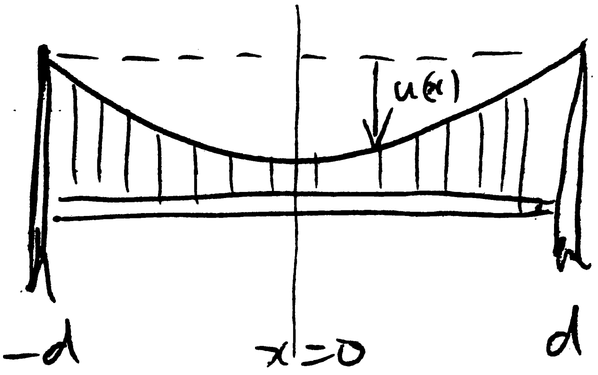
$$\underline{n} \times \left(\underline{n} \times \iint \nabla \phi \, dA \right) = \underline{n} \times \oint \phi \, d\underline{l}$$

$$\therefore \quad (\underline{n} \cdot \underline{n}) \iint \nabla \phi \, dA - \left(\underline{n} \cdot \iint \nabla \phi \, dA \right) \underline{n} = \underline{n} \times \oint \phi \, d\underline{l}$$

But $\underline{n} \cdot \underline{n} = 1$, and $\nabla \phi \perp \underline{n}$ for 2D field ϕ, so $\underline{n} \cdot \iint \nabla \phi \, dA =$

$$\therefore \quad \iint \nabla \phi \, dA = \underline{n} \times \oint \phi \, d\underline{l}$$

4. (a)



In order for there to be no shear force or bending moment in the bridge deck, each element of bridge must be supported in equilibrium by its own cables. In other words, the weight is transferred directly to the main suspension cable. So this cable hangs in equilibrium with a mass distribution which is proportional to x

To lift each element to the datum line would require potential energy $(m\,\delta x)\,g\,u$ to be expended. So total potential energy $V = -\int_{-d}^{d} mgu\,dx$

Equilibrium for minimum V, ie maximum $\int_{-d}^{d} mgu\,dx$

Constraint is $L = \int ds = \int_{-d}^{d} \sqrt{1+u'^2}\,dx$ ①

(b) So introduce Lagrange multiplier λ, and seek to make $\int \left\{ u + \lambda\sqrt{1+u'^2} \right\} dx$ stationary.

Euler-Lagrange equation: $\frac{d}{dx}\left(\lambda \cdot \frac{1}{2}\left(1+u'^2\right)^{-1/2} \cdot 2u'\right) = 1$

$$\therefore \frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = \frac{1}{\lambda}$$

$$\therefore \frac{u'}{\sqrt{1+u'^2}} = \frac{x}{\lambda} + A \text{, say}$$

Symmetry: $x \to -x$ means $u' \to -u'$, so $A = 0$

4 contd.

$$\therefore \quad u'^2 = (x/\lambda)^2(1+u'^2)$$

$$\therefore u'^2 = \frac{(x/\lambda)^2}{1-(x/\lambda)^2}$$

$$\therefore u = \int \frac{x/\lambda \, dx}{\sqrt{1-(x/\lambda)^2}} . \quad \text{Set } x = \lambda \sin\theta,\ dx = \lambda\cos\theta\, d\theta$$

$$= \int \frac{\sin\theta \cdot \lambda\cos\theta\, d\theta}{\cos\theta} = \lambda \int \sin\theta\, d\theta$$

$$= -\lambda\cos\theta + B \quad \text{say}$$

$$\therefore \lambda^2\sin^2\theta + \lambda^2\cos^2\theta = x^2 + (u-B)^2 = \lambda^2$$

This is the equation of a circle.

$u=0$ at $x=0$ means $B = \lambda$

Value of λ is determined by the constraint (1)