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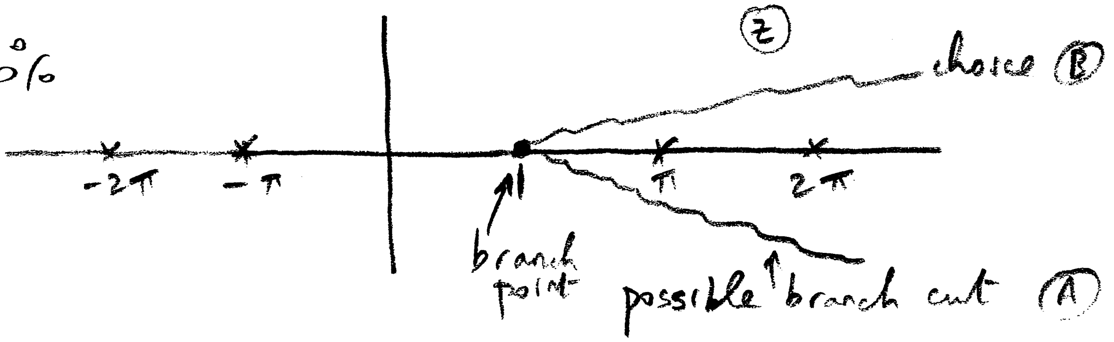
1. (a) (i) $f(z) = \dfrac{z\sqrt{z-1}}{\sin z}$

Removeable singularity at $z = 0$.
$\sin z$ has zeros at $z = n\pi$ where n = non-zero integer.
Near $z = n\pi$ write $z = n\pi + (z - n\pi)$
$\sin z = \sin(n\pi + (z - n\pi)) = (z - n\pi)\cos n\pi$
Now $\cos n\pi = (-1)^n$

At poles we have $f(z) \sim \dfrac{n\pi\,(n\pi - 1)^{1/2}}{(-1)^n\,(z - n\pi)}$

So the residue at $z = n\pi$ is $\underline{n\pi\,(n\pi-1)^{1/2}(-1)^n}$

20%

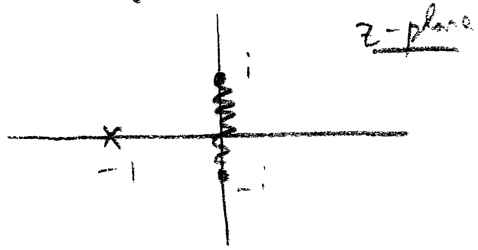


For choice (A) of branch cut, residue $= (-1)^n\, n\pi\sqrt{n\pi - 1}$

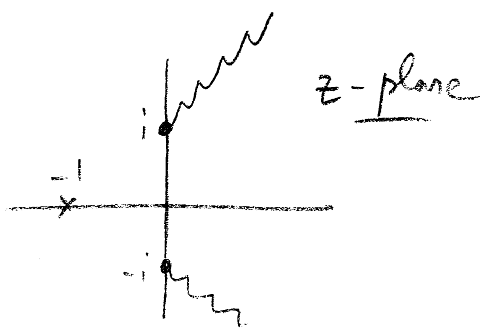
For choice (B), residue $= (-1)^{n+1}\, n\pi\sqrt{n\pi - 1}$
In each case, $n = \pm 1, \pm 2, \ldots$

(ii) $f(z) = \dfrac{\sqrt{z^2+1}}{(z+1)^3} = \dfrac{(z+i)^{1/2}(z-i)^{1/2}}{(z+1)^3}$

Branch points at $z = \pm i$.
Pole of order 3 at $z = -1$.



or



1. (a) (ii) contd.

Taylor series expansion of $(z^2+1)^{1/2}$ about the point $z=-1$.

$$g(z) = (z^2+1)^{\frac{1}{2}} = g(-1) + (z+1)\,g'(-1) + \frac{(z+1)^2}{2!}\,g''(-1) + \ldots$$

So, the residue at $z=-1$ is $\frac{g''(-1)}{2!}$

Now $g'(z) = \frac{1}{2}(z^2+1)^{-\frac{1}{2}} \cdot 2z = z\,(z^2+1)^{-\frac{1}{2}}$

$$\Rightarrow g''(z) = (z^2+1)^{-\frac{1}{2}} - \frac{1}{2}\,z\,(z^2+1)^{-3/2} \cdot 2z$$

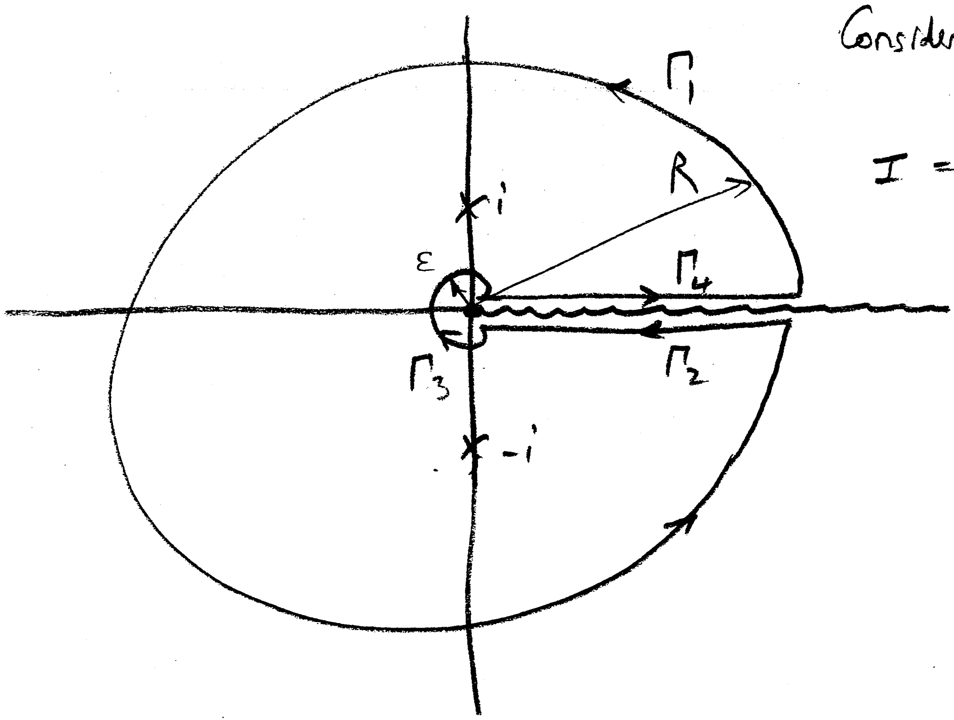
$$\Rightarrow g''(z) = (z^2+1)^{-\frac{1}{2}} - z^2\,(z^2+1)^{-3/2}$$

$$\Rightarrow g''(z=-1) = 2^{-\frac{1}{2}} - 2^{-3/2} = \frac{1}{\sqrt{2}}\left(1-\frac{1}{2}\right) = \frac{1}{2\sqrt{2}}$$

So, the residue at $z=-1$ is $\frac{1}{4\sqrt{2}}$.

[40%]

1 (b) $I = \int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx = ?$



Consider $J \equiv \oint \frac{z^{1/3}}{z^2+1}\,dz$

$I = \int_{\Gamma_4} \frac{z^{1/3}}{z^2+1}\,dz$

Now, $\int_{\Gamma_1} \frac{z^{1/3}}{z^2+1}\,dz \to 0$ as $R \to \infty$

$\int_{\Gamma_3} \frac{z^{1/3}}{z^2+1}\,dz \to 0$ as $\varepsilon \to 0$

Now $\int_{\Gamma_2} = -\int_0^\infty \frac{e^{i2\pi/3}\,x^{1/3}}{x^2+1}\,dx = -e^{i2\pi/3}\,I$

So, $J = \left(1 - e^{i2\pi/3}\right) I = 2\pi i \times$ (Sum of residues)

Residues of $\frac{z^{1/3}}{z^2+1} = ?$

$\frac{z^{1/3}}{z^2+1} = \frac{z^{1/3}}{(z+i)(z-i)}$ At $z = i = e^{i\pi/2}$

At $z = i = e^{i\pi/2}$, residue $= \frac{e^{i\pi/6}}{2i}$

At $z = -i = e^{i3\pi/2}$, residue $= \frac{-e^{i\pi/2}}{2i}$

So,
$$J = \frac{2\pi i}{2i} \times \left(e^{i\pi/6} - e^{i\pi/2}\right)$$
$$= \pi\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} - i\right)$$
$$\Rightarrow \quad J = \pi\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

Also,
$$J = \left(1 - e^{i2\pi/3}\right)I = \left(1 - \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)I$$
$$= \left(1 + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)I = \sqrt{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)I$$

Equate,
$$\Rightarrow \quad I = \frac{\pi}{\sqrt{3}}$$

[60%]

2. (a) Laurent expansion of $f(z) = \dfrac{\cos z}{z^2(z-2)}$

<u>Simple pole at $z = 2$</u>

write $f(z) = \dfrac{g(z)}{z-2}$ where $g(z) = z^{-2}\cos z$

Expand $g(z)$ as a Taylor series about $z = 2$.

$$g(z) \sim g(z=2) + (z-2)\, g'(z=2) + \dots$$

$$g(z=2) = \frac{\cos 2}{4} \qquad g'(z) = -2z^{-3}\cos z - z^{-2}\sin z$$

$$\Rightarrow g'(z=2) = -\frac{1}{4}\cos 2 - \frac{1}{4}\sin 2 = -\frac{1}{4}(\cos 2 + \sin 2)$$

$$\Rightarrow f(z) \approx \frac{1}{4}\frac{\cos 2}{z-2} - \frac{1}{4}(\cos 2 + \sin 2) + \dots$$

about $z = 2$.

<u>Pole of order 2 at $z = 0$</u>

Now $\cos z \approx 1 - \frac{1}{2}z^2 + \dots$

$$\frac{1}{z-2} = (z-2)^{-1} = \frac{1}{-2}\left(1 - \tfrac{1}{2}z\right)^{-1}$$

$$= -\frac{1}{2}\left(1 + \tfrac{1}{2}z + \frac{(-1)(-2)}{2!}\left(-\tfrac{1}{2}z\right)^2 + \dots\right)$$

$$= -\frac{1}{2}\left(1 + \tfrac{1}{2}z + \tfrac{1}{4}z^2 + \dots\right)$$

$$\Rightarrow f(z) \approx -\frac{1}{2}\frac{1}{z^2}\left(1 - \tfrac{1}{2}z^2 + \dots\right)\left(1 + \tfrac{1}{2}z + \tfrac{1}{4}z^2 + \dots\right)$$

$$= -\frac{1}{2z^2}\left(1 - \tfrac{1}{2}z^2 + \dots + \tfrac{1}{2}z + \dots\right)$$

$$\Rightarrow f(z) = -\frac{1}{2z^2} - \frac{1}{4z} + \frac{1}{4} + \dots$$

[30%]

2. (b) (i) $F(\omega) = \int_0^\infty e^{-ax} e^{i\omega x} \, dx$

$$= \int_0^\infty e^{(i\omega - a)x} \, dx = \frac{-1}{i\omega - a} = \underline{\underline{\frac{1}{a - i\omega}}}$$

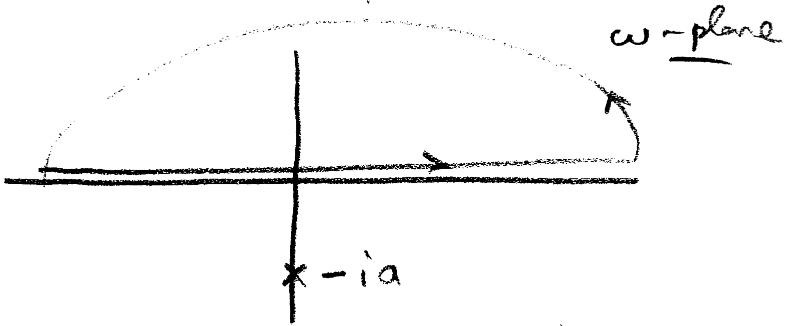
(ii) $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \, d\omega$

Consider Jordan's Lemma.
If $x < 0$, close the contour in the U.H.P.

Note ~~that~~ $F(\omega) = \dfrac{i}{\omega + ia}$

so $F(\omega)$ has a simple pole at $\omega = -ia$.

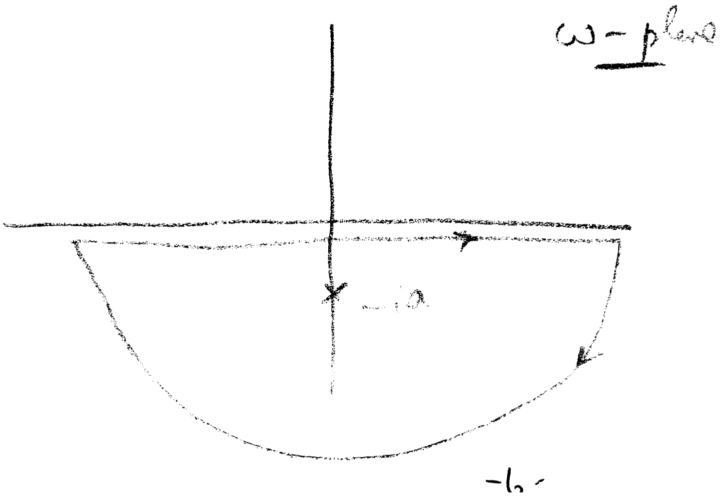
<ins>Case $x < 0$</ins>



So, $f(x) = \frac{1}{2\pi} \oint F(\omega) e^{-i\omega x} \, d\omega = 0$

residue theorem

<ins>Now consider $x > 0$</ins>



In the evaluation of $\frac{1}{2\pi} \oint F(\omega) e^{-i\omega x} \, d\omega$ close the contour in the L.H.P.

2 $f(x) = \frac{1}{2\pi} \oint \frac{i}{\omega + ia} e^{-i\omega x} \, d\omega \quad , \quad x > 0$

$= \frac{-1}{2\pi} \times 2\pi i \times \text{Residue}$

Residue of $\frac{i}{\omega+ia} e^{-i\omega x}$ at $\omega = -ia$ is

Residue $= i \, e^{-ax}$

$\Rightarrow \quad f(x) = e^{-ax} \quad , \quad x > 0.$

[70%]

SECTION B

Q3

(a) The objective function (to be minimized) is

$$f(\underline{x}) = -0.30x_1 - 0.34x_2 - 0.37x_3 - 0.39x_4$$

subject to (use of regular)

$$x_1 + 0.9x_2 + 0.8x_3 + 0.7x_4 = 64\times10^3$$

and (use of octane)

$$0.1x_2 + 0.2x_3 + 0.3x_4 = 12\times10^3$$

[10%]

(b) The initial tableau is

$$\begin{bmatrix} 1 & 0.9 & 0.8 & 0.7 & 64\times10^3 \\ 0 & 0.1 & 0.2 & 0.3 & 12\times10^3 \\ -0.30 & -0.34 & -0.37 & -0.39 & 0 \end{bmatrix} \begin{array}{l} \text{Row1} \\ \text{Row2} \\ \text{Row3} \end{array}$$

For the given initial feasible solution, x_1 and x_4 are the initial basic variables, hence in canonical form the initial tableau is

$$\begin{array}{ccccc} \Downarrow & & & \Downarrow & \\ \end{array}$$
$$\begin{bmatrix} 1 & 2/3 & 1/3 & 0 & 36\times10^3 \\ 0 & 1/3 & 2/3 & 1 & 40\times10^3 \\ 0 & -0.01 & -0.01 & 0 & 26.4\times10^3 \end{bmatrix} \begin{array}{l} \text{Row4} = \text{Row1} - (7/3)\text{Row2} \\ \text{Row5} = (10/3)\text{Row2} \\ \text{Row6} = \text{Row3} + 0.3\text{Row4} + 0.39\text{Row5} \end{array}$$

Here there is a choice of entering variable, the reduced costs for x_2 and x_3 are the same (–0.01). Choosing x_2, the leaving variable is x_1, $36\times10^3\div(2/3)$ being less than $40\times10^3\div(1/3)$. Hence the next tableau (in canonical form) is

$$\begin{array}{ccccc} & \Downarrow & & \Downarrow & \\ \end{array}$$
$$\begin{bmatrix} 3/2 & 1 & 1/2 & 0 & 54\times10^3 \\ -1/2 & 0 & 1/2 & 1 & 22\times10^3 \\ 0.015 & 0 & -0.005 & 0 & 26.94\times10^3 \end{bmatrix} \begin{array}{l} \text{Row7} = (3/2)\text{Row4} \\ \text{Row8} = \text{Row5} - (1/2)\text{Row4} \\ \text{Row9} = \text{Row6} + 0.01\text{Row7} \end{array}$$

Now x_3 is the entering variable (the only –ve reduced cost) and x_4 is the leaving variable, $22\times10^3\div(1/2)$ being less than $54\times10^3\div(1/2)$. Hence the next tableau (in canonical form) is

$$\begin{array}{ccccc} & \Downarrow & \Downarrow & & \\ \end{array}$$
$$\begin{bmatrix} 2 & 1 & 0 & -1 & 32\times10^3 \\ -1 & 0 & 1 & 2 & 44\times10^3 \\ 0.010 & 0 & 0 & 0.010 & 27.16\times10^3 \end{bmatrix} \begin{array}{l} \text{Row10} = \text{Row7} - \text{Row8} \\ \text{Row11} = 2\text{Row8} \\ \text{Row12} = \text{Row9} + 0.005\text{Row11} \end{array}$$

There are now no –ve reduced costs so this is the optimum.

The optimal balance of fuel production is $\underline{x} = (0, 32\times10^3, 44\times10^3, 0)$.

[70%]

(c) In phase 1 of the Simplex method additional variables (one per constraint equation) are introduced and a new objective function is defined. In this case the new constraint equations are

$$x_1 + 0.9x_2 + 0.8x_3 + 0.7x_4 + x_5 = 64\times10^3$$

and

$$0.1x_2 + 0.2x_3 + 0.3x_4 + x_6 = 12\times10^3$$

and the new objective function (to be minimized) is

$$f(\underline{x}) = x_5 + x_6$$

Hence the initial Simplex tableau is

$$\begin{array}{cl} \left[\begin{array}{ccccccc} 1 & 0.9 & 0.8 & 0.7 & 1 & 0 & 64\times10^3 \\ 0 & 0.1 & 0.2 & 0.3 & 0 & 1 & 12\times10^3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] & \begin{array}{l} \text{Row1} \\ \text{Row2} \\ \text{Row3} \end{array} \end{array}$$

The additional variables (x_5 and x_6) are the initial basic variables.

Solving this problem will yield a solution with $x_5 = x_6 = 0$ and $f(\underline{x}) = 0$ which satisfies the original constraint equations, thus giving a feasible initial solution for the linear programming problem.

[20%]

Q4

(a) The task is to minimise $f = -I = \dfrac{-31.5r^2n^2}{6r+9l+10t}$

subject to $2\pi rlt - V = 0$

and $2\pi rn - L = 0$

[10%]

(b) The Lagrangian is $\ell = \dfrac{-31.5r^2n^2}{6r+9l+10t} + \lambda_1(2\pi rlt - V) + \lambda_2(2\pi rn - L)$

Therefore, the first-order optimality conditions are

$$\frac{\partial \ell}{\partial r} = \frac{-63rn^2(6r+9l+10t)+189r^2n^2}{(6r+9l+10t)^2} + \lambda_1 2\pi lt + \lambda_2 2\pi n = 0 \qquad (1)$$

$$\frac{\partial \ell}{\partial l} = \frac{283.5r^2n^2}{(6r+9l+10t)^2} + \lambda_1 2\pi rt = 0 \qquad (2)$$

$$\frac{\partial \ell}{\partial t} = \frac{315r^2n^2}{(6r+9l+10t)^2} + \lambda_1 2\pi rl = 0 \qquad (3)$$

$$\frac{\partial \ell}{\partial n} = \frac{-63r^2n}{(6r+9l+10t)} + \lambda_2 2\pi r = 0 \qquad (4)$$

$$2\pi rlt - V = 0 \qquad (5)$$

$$2\pi rn - L = 0 \qquad (6)$$

(2)÷(3) ⇒ $\dfrac{t}{l} = \dfrac{283.5}{315} = 0.9 \;\Rightarrow\; \underline{\underline{t = 0.9l}}$

From (4) $$2\pi\lambda_2 = \frac{63rn}{(6r+9l+10t)} \qquad (7)$$

From (2) $$2\pi\lambda_1 t = \frac{-283.5rn^2}{(6r+9l+10t)^2} \qquad (8)$$

Substituting in (1) using (7) and (8)

$$\frac{-63rn^2(6r+9l+10t)+189r^2n^2}{(6r+9l+10t)^2} - \frac{283.5lrn^2}{(6r+9l+10t)^2} + \frac{63rn^2}{(6r+9l+10t)} = 0$$

$$\therefore \quad \cancel{\frac{-63rn^2}{(6r+9l+10t)}} + \frac{189r^2n^2}{(6r+9l+10t)^2} - \frac{283.5lrn^2}{(6r+9l+10t)^2} + \cancel{\frac{63rn^2}{(6r+9l+10t)}} = 0$$

$$\therefore \quad 189r^2n^2 = 283.5lrn^2 \Rightarrow r = \frac{283.5}{189}l \Rightarrow \underline{\underline{r = 1.5l}}$$

Using these results in (5)

$$2\pi rlt = 2\pi(1.5l)l(0.9l) = 2.7\pi l^3 = V = \frac{\pi}{7290} \Rightarrow l^3 = \frac{1}{19683} \Rightarrow \underline{\underline{l = 1/27\text{ m}}}$$

and hence $$\underline{\underline{r = 1.5l = 1/18\text{ m}}}$$

$$\underline{\underline{t = 0.9l = 1/30\text{ m}}}$$

Using (6) $$n = \frac{L}{2\pi r} = \frac{2\pi}{2\pi(1/18)} \Rightarrow \underline{\underline{n = 18\text{ turns}}}$$

and so $$I = \frac{31.5r^2n^2}{6r+9l+10t} = \frac{31.5\times[1/18]^2\times[18]^2}{(6\times[1/18]+9\times[1/27]+10\times[1/30])} \Rightarrow \underline{\underline{I = 31.5\ \mu\text{H}}}$$

[70%]

(c) The sensitivity of the objective to the constraints is given by the values of the Lagrange multipliers.

From (8) $$\lambda_1 = -\frac{283.5rn^2}{2\pi t(6r+9l+10t)^2}$$

and from (7) $$\lambda_2 = \frac{63rn}{2\pi(6r+9l+10t)}$$

From part (b) we know that, for the values specified, $(6r+9l+10t)=1$

$$\therefore \quad \lambda_1 = -\frac{283.5\times[1/18]\times[18]^2}{2\pi\times[1/30]} = -24365$$

and $$\lambda_2 = \frac{63\times[1/18]\times[18]}{2\pi} = 10.03$$

Thus, increasing V by 1% changes I by $0.01\lambda_1 V = -0.01\times 24365\times\frac{\pi}{7290} = \underline{\underline{-0.105\ \mu\text{H}}}$

and increasing L by 1% changes I by $0.01\lambda_2 L = 0.01\times 10.03\times 2\pi = \underline{0.63\ \mu\text{H}}$

[20%]