ENGINEERING TRIPOS PART IIB

Tuesday 26 April 2005 9 to 10.30

Module 4A8

ENVIRONMENTAL FLUID MECHANICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A8 Data Card (5 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) Show that the Dry Adiabatic Lapse Rate (DALR) is given by

$$\left.\frac{dT}{dz}\right|_{DALR} = -\frac{g}{c_p},$$

where the symbols have their usual meaning. [20%]

(b) Explain what is meant by the Brunt-Vaisala frequency. [10%]

(c) Two thermometers are located in a vertical line at heights h_1 and h_2 ($h_2 > h_1$) from the ground a small distance upwind from a large ridge of length L. The two thermometers read temperatures T_1 and T_2 respectively and in a clear night, it is found that $T_2 > T_1$. You may assume that the temperature gradient normal to the surface is linear.

(i) Find an expression for the wind speed U that will lead to the maximum generation of internal waves. [50%]

(ii) Assuming a logarithmic velocity profile in the boundary layer approaching the two thermometers, find an expression for the height at which the Richardson number first exceeds ¼. What is the significance of this value? You may assume that the wall shear velocity $u^* = \alpha U$. [20%]

2 A mixing vessel is a cylinder of diameter 1 m and height 1 m and is stirred by an impeller of size equal to the diameter of the vessel. The purpose of the mixing is to uniformly disperse solid polymer particles in water. At steady operation, the mixer is expected to produce a smallest eddying motion of 0.1 mm so that the polymer chains are not broken by too severe straining.

(a) Estimate the power required to drive the impeller at steady state ignoring mechanical losses and the presence of the solid particles and assuming that all energy input goes into the turbulent motion of the water. You may treat the turbulence in the vessel as fully homogeneous and take the kinematic viscosity of water as $10^{-6}\,\mathrm{m^2\,s^{-1}}$. [30%]

(b) If the largest eddying motion is approximately equal to the vessel's diameter, estimate the kinetic energy per unit mass of the fluid in the tank. What is the Reynolds number of the turbulence and the large-eddy timescale? [30%]

(c) If the impeller is switched off, estimate the time elapsed until the smallest eddying motion in the tank becomes equal to 0.4 mm. Assume that the largest eddy size remains unaltered. [40%]

3 The concentration (in μg m^{-3}) c of chemicals A and B in the air above a city are homogeneous in the mean, but finite small-scale fluctuations exist. The pollutants react according to the reaction A+A⇒B and the rate of generation of B (in μg m^{-3} s^{-1}) is given by $\dot{w}_B = k\, c_A c_A$, where k is a constant. Turbulence is assumed to be homogeneous with characteristic timescale T_{turb}. Initially, species A has a mean concentration $\bar{c}_A = 1$ and a variance $\sigma_A^2 = 1$. For B, initially $\bar{c}_B = \sigma_B^2 = 0$.

(a) Neglecting the species concentration fluctuations, derive expressions for the means $\bar{c}_A$ and $\bar{c}_B$ as a function of time. [60%]

(b) Consider the equation for the variance of A and solve it by neglecting the chemistry source term. [20%]

(c) Write down a differential equation for the evolution of $\bar{c}_A$ as a function of time *including* fluctuations and, without solving it, discuss how different is its solution to that of part (a). Explain what happens as $k\, T_{turb}$ becomes very small. [20%]

4 (a) A chimney stack of height 100 m emits 10^{-6} kg s^{-1} of inert pollutant in a uniform wind of speed 3 m s^{-1}. For the first 2 km downwind of the stack, rural conditions prevail so that the dispersion coefficient σ (assumed uniform in all directions) is equal to $0.02x$, where x is the distance from the stack. Following this, urban conditions prevail with $\sigma = 0.1x$. Find the maximum ground concentration at a distance of 4 km from the stack. [70%]

(b) Why is dispersion above a city more vigorous than that above a flat field? [30%]

END OF PAPER

4A8: Environmental Fluid Mechanics

Part I: Turbulence and Fluid Mechanics

DATA CARD

Rotating Flows

Geostrophic Flow

$$-\frac{1}{\rho}\nabla p = 2\underline{\Omega}\times\underline{u}$$

Ekman Layer Flow

$$-2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

OR

$$-2\Omega_z v = \nu\frac{\partial^2 u}{\partial z^2}$$

$$-2\Omega_z v(u_0 - u) = \nu\frac{\partial^2 v}{\partial z^2}$$

GEOSTROPHIC VELOCITY

Solution is

$$u = u_0\left[1 - e^{-z/\Delta}\cos\frac{z}{\Delta}\right]$$

$$v = u_0 e^{-z/\Delta}\sin\frac{z}{\Delta}$$

$$\Delta = \left(\frac{\nu}{\Omega_z}\right)^{1/2}$$

Turbulent Flows – Incompressible

Continuity Equation $$\nabla \bullet \underline{U} = \frac{\partial U_i}{\partial x_i} = 0$$

Momentum Equation $$\rho \frac{D\underline{U}}{Dt} = -\nabla P + \mu \nabla^2 \underline{U} + \underline{F}$$

$$\rho \frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2} + F_i$$

Enthalpy Equation $$\rho c_p \frac{DT}{Dt} = -k \frac{\partial^2 T}{\partial x_i^2}$$

Reynolds Transformation $$U_i = \overline{U_i} + u_i \quad \text{etc}$$

Reynolds Stress $$= -\rho \overline{u_i u_j}$$

Reynolds Heat Flux $$= -\overline{\rho c_p u_j \Theta}$$

Turbulent Kinetic Energy Equation

$$\frac{D}{Dt}\frac{q^2}{2} = -\overline{u_i u_k}\frac{\partial \overline{U_i}}{\partial x_k} - \nu \overline{\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_k}\right)} + \frac{\overline{f_i u_i}}{\rho} + \text{ transport of kinetic energy forms}$$

In flows with thermally driven motion

$$\frac{f_i u_i}{\rho} = \frac{g}{T} \bullet \overline{\Theta u_i}$$

i = Vertical direction

Dissipation of turbulent kinetic energy $$\varepsilon \approx \frac{u'^3}{\ell}$$

Kolmogorov microscale $$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

Taylor microscale (λ) $$\varepsilon = 15\nu \frac{u'^2}{\lambda^2}$$

(ν is the kinematic viscocity)

Density Influenced Flows

Atmospheric Boundary Layer

$$\left.\frac{dT}{dz}\right|_{\text{NEUTRAL STABILITY}} = -\frac{g}{c_p} = \left.\frac{dT}{dz}\right|_{\text{DALR}}$$

$$R_i = \frac{g}{T}\frac{\dfrac{dT}{dz} - \left.\dfrac{dT}{dz}\right|_{\text{DALR}}}{\left(\dfrac{dU}{dz}\right)^2} = \text{RICHARDSON NUMBER}$$

Neutral Stability

$$U = \frac{u_*}{\kappa}\ln\frac{z}{z_0}; \quad \frac{dU}{dz} = \frac{u_*}{\kappa z}$$

$$u_* = \sqrt{\frac{\tau_w}{\rho}} \; ; \quad \kappa = \text{von Karman Constant} = 0.40$$

Non-Neutral Stability

$$\text{L} = \text{Monin-Obukhov length} = -\frac{u_*^3}{\kappa \dfrac{g}{T}\dfrac{Q}{\rho c_p}}$$

Q = surface heat flux

$$\frac{dU}{dz} = \frac{u_*}{\kappa z}\left(1 - 15\frac{z}{L}\right)^{-1/4} \quad \text{Unstable}$$

$$= \frac{u_*}{\kappa z}\left(1 + 4.7\frac{z}{L}\right) \quad \text{Stable}$$

Buoyant plume for a point source

$$\frac{d}{dz}\pi R^2 w = 2\pi R u_e \qquad \text{(i)}$$

$$\frac{d}{dz}\rho\pi R^2 w = \rho_a 2\pi R u_e \qquad \text{(ii)}$$

$$\frac{d}{dz}\rho\pi R^2 w = g(\rho_a - \rho)\pi R^2 \qquad \text{(iii)}$$

(i) and (ii) give

$$\pi R^2 w\left(\frac{\rho_a - \rho}{\rho_a}\right)g = \text{constant} = F_0 \text{ (buoyancy flux)}$$

$$u_e = \alpha w$$

(α = Entrainment coefficient)

<u>Brunt – Vaisala Frequency</u>

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz} = \frac{g}{T}\frac{dT}{dz}$$

Actually,

$$N^2 = \frac{g}{T}\left(\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{\text{DALR}}\right)$$

Brunt – Vaisala Frequency is also called Buoyancy Frequency

4A8: Environmental Fluid Mechanics

Part II: Dispersion of Pollution in the Atmospheric Environment

DATA CARD

Transport equation for the mean of the reactive scalar ϕ:

$$\frac{\partial \bar{\phi}}{\partial t} + \bar{u}_j \frac{\partial \bar{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K \frac{\partial \bar{\phi}}{\partial x_j} \right) + \bar{w}$$

Transport equation for the variance of the reactive scalar ϕ:

$$\frac{\partial g}{\partial t} + \bar{u}_j \frac{\partial g}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K \frac{\partial g}{\partial x_j} \right) + 2K\left(\frac{\partial \bar{\phi}}{\partial x_j} \right)^2 - \frac{2}{T_{turb}} g + 2\overline{\phi' w'}$$

Mean concentration of pollutant after instantaneous release of Q kg at t=0:

$$\bar{\phi}(x,y,z,t) = \frac{Q}{8(\pi t)^{3/2}(K_x K_y K_z)^{1/2}} \exp\left[-\frac{1}{4t}\left(\frac{(x-x_0)^2}{K_x} + \frac{(y-y_0)^2}{K_y} + \frac{(z-z_0)^2}{K_z} \right) \right]$$

Gaussian plume spreading in two dimensions from a source at $(0,0,z_0)$ emitting Q kg/s:

$$\bar{\phi}(x,y,z) = \frac{Q}{2\pi} \frac{1}{U\sigma_y \sigma_z} \exp\left[-\left(\frac{y^2}{2\sigma_y^2} + \frac{(z-z_0)^2}{2\sigma_z^2} \right) \right]$$

One-dimensional spreading from line source emitting Q/L kg/s/m :

$$\bar{\phi}(x,y) = \frac{Q}{UL} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left(-\frac{y^2}{2\sigma_y^2} \right)$$

Relationship between eddy diffusivity and dispersion coefficient:

$$\sigma^2 = 2\frac{x}{U} K$$