ENGINEERING TRIPOS PART IIB

Wednesday 4^{th} May 2005 2.30 to 4.00

Module 4A10

FLOW INSTABILITY

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A10 Data sheet (2 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 In deep glacial lochs, it is possible for fresh water to be moving out to sea while the tide simultaneously draws seawater inland. This leads to the situation shown in figure 1.

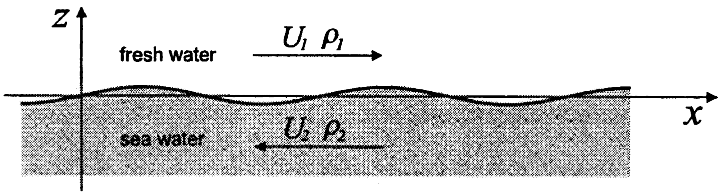

Figure 1

The velocity potentials of the flows are:

$$\begin{aligned}\phi_1(x,z,t) &= U_1 x + f_1(z)e^{(st+ikx)}\\ \phi_2(x,z,t) &= U_2 x + f_2(z)e^{(st+ikx)}\end{aligned}$$

where $f_1(z) = A_1 e^{-kz} + B_1 e^{kz}$, $f_2(z) = A_2 e^{-kz} + B_2 e^{kz}$ and U_2 is negative. The interface is given by the line $\eta = \eta_0 e^{(st+ikx)}$. The kinematic boundary conditions at the interface are:

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x} \qquad \text{and} \qquad \frac{\partial \phi_2}{\partial z} = \frac{\partial \eta}{\partial t} + U_2 \frac{\partial \eta}{\partial x}$$

(a) Assume that the layers of fresh water and seawater are very thick so that $f_1(\infty) \to 0$ and $f_2(-\infty) \to 0$. For small perturbations of the mean flow, show that these two boundary conditions require that: [20%]

$$\frac{A_1}{B_2} = -\frac{(s + ikU_1)}{(s + ikU_2)}$$

(b) By equating the pressures at the interface, find the dynamic boundary condition. This is another expression for A_1/B_2, which must also be satisfied at the interface. [50%]

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(c) You may assume without proof that combining these boundary conditions produces the following expression for s/k:

$$\frac{s}{k} = -i\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \sqrt{\frac{\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1+\rho_2)^2} - \frac{g(\rho_2 - \rho_1)}{k(\rho_2+\rho_1)}} \; .$$

Find an expression for the unstable wavenumbers, k, in terms of g, ρ_1, ρ_2 and $(U_1 - U_2)$. Comment on this result. [10%]

(d) In reality, the shear layer between the fresh water and the seawater will have a finite thickness. This will stabilise short wavelength perturbations. Consider the loch shown in figure 2. The seawater speeds up as it passes over the lip where the loch meets the sea. Further out to sea there is an oil platform whose buoyancy tanks rest at the interface between the fresh water and the seawater. Describe what will happen to the oil platform when the flow is unstable at the lip but stable at the oil platform. Justify your answer using the expression for s/k given above. [20%]

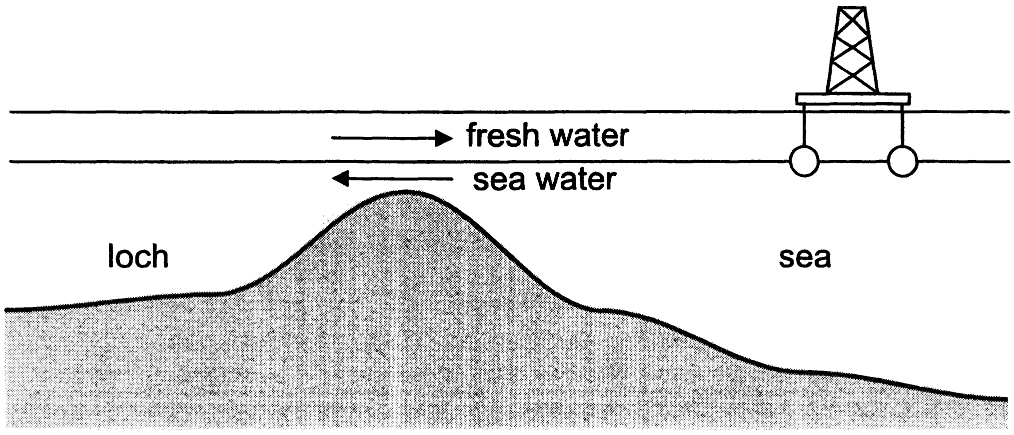

Figure 2

2 A rotating fluid with uniform density has azimuthal velocity profile $v(r)$ and no axial motion.

(a) Consider a ring of fluid at radius r_1 and a ring of fluid at radius r_2. Both rings have mass δm. What is the angular momentum and kinetic energy of each ring? [10%]

(b) The two rings swap places, conserving their individual angular momenta. What is the final kinetic energy? [10%]

(c) The circulation Γ is equal to the product $2\pi r v$. Show that this flow will be stable when $\Gamma_2^2 > \Gamma_1^2$: [20%]

The same fluid is now given axial velocity profile $u(r)$ in addition to the original azimuthal velocity profile $v(r)$. As before, consider rings of fluid at radius r_1 and r_2, each of mass δm.

(d) What is the kinetic energy of each ring? [10%]

(e) The two rings swap places, conserving their individual angular momenta. In swapping places, they exchange some axial momentum and end up with identical axial velocities. Deduce a criterion for the stability of this flow in terms of $(\Gamma_2^2 - \Gamma_1^2)$ and $(u_2 - u_1)$. Is the axial velocity difference stabilising or destabilising? [40%]

(f) Instabilities and turbulence must be avoided in blood vessels because they provoke clotting. The Reynolds number in main arteries is around 2000, which is near to the point of turbulent transition. A company manufactures plastic tubing which is used in arterial grafts. Can you suggest a way to design the tube such that it delays the transition to turbulence within the graft? [10%]

3 A tube contains fluid moving at constant velocity U. It has infinite extent in the x-direction and is constrained to move in the y-direction only. The equation of motion of the tube, ignoring gravity, is:

$$EI\frac{\partial^4 Y}{\partial x^4} + (pA + \rho AU^2 - T)\frac{\partial^2 Y}{\partial x^2} + (\rho A + m)\frac{\partial^2 Y}{\partial t^2} + 2\rho AU\frac{\partial^2 Y}{\partial t \partial x} = 0$$

where Y is the deflection of the centreline in the y-direction, EI is the flexural rigidity, T is the tension in the pipe, p is the internal gauge pressure, ρ is the fluid density, m is the mass per unit length and A is the cross-sectional area.

(a) What is the term proportional to U^2 called? How does it arise and what effect does it have on sinuous oscillations of the pipe? [15%]

(b) What is the term proportional to U called? How does it arise and what effect does it have on sinuous oscillations of the pipe? [15%]

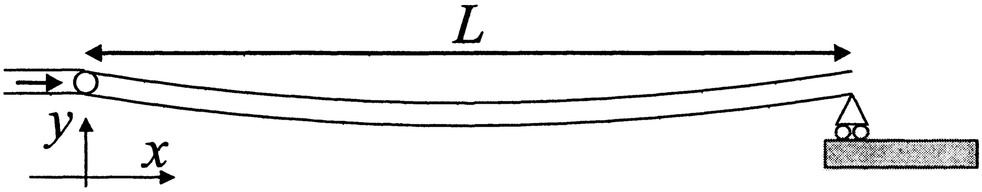

Figure 3

An experiment is performed on a flexible tube, as shown in figure 3. Both ends of the tube can pivot freely. The end at $x = 0$ is constrained to have zero x- and y-displacement. The end at $x = L$ is constrained to have zero y-displacement but can move without friction in the x-direction. A flow passes through the pipe at velocity U. Assume that both gravity and friction within the pipe can be neglected.

(c) To a first approximation, the term proportional to U in the above equation of motion can be neglected. By considering disturbances of the form $Y = Y_0 e^{st} \sin kx$, develop an expression for the flow velocity at which the pipe will buckle. [45%]

(d) A nozzle is put onto the end of the pipe, halving the flow area. What is the flow velocity at which the pipe will buckle? [25%]

4 A boat is moored by its front, facing into a steady wind, as shown in figure 4. Its centre of mass is behind its aerodynamic centre and it is free to move from side to side and to rotate. The water is still.

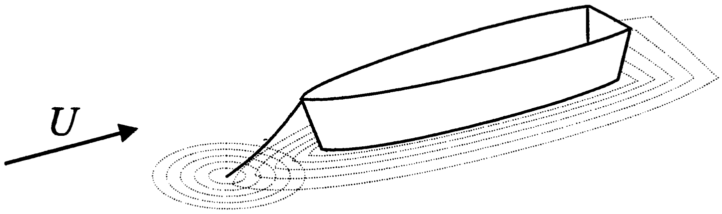

Figure 4

(a) Describe the motion of the boat when the wind changes direction slightly. No calculations are required. [20%]

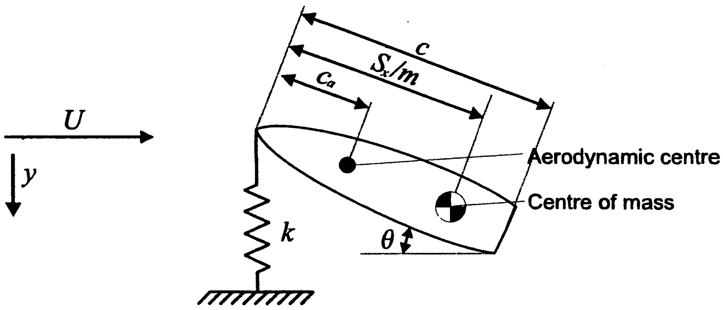

Figure 5: Boat viewed from above

The mooring rope can be modelled as a spring, as shown in figure 5. Ignoring motion of the boat in the wind's direction and ignoring damping terms, the equations of motion of the boat are given by:

$$m\ddot{y} + S_x\ddot{\theta} + ky = F_y$$
$$I\ddot{\theta} + S_x\ddot{y} = F_\theta$$

where m is the mass of the boat and I is the boat's moment of inertia about the point where it is attached to the rope.

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The force and moment due to the wind can be approximated by:

$$\begin{aligned} F_y &= -q\theta \\ F_\theta &= -qc_a\theta \end{aligned}$$

where q is positive and is given by:

$$q = \frac{1}{2}\rho U^2 c \left.\frac{\partial C_L}{\partial \alpha}\right|_{\alpha=0}$$

(b) Assuming solutions of the form $y = Y_0 e^{st}$ and $\theta = \theta_0 e^{st}$, the equations of motion can then be expressed in the matrix form shown below. Find M. [20%]

$$\mathrm{M}\begin{pmatrix} Y_0 \\ \theta_0 \end{pmatrix} = 0$$

(c) A trivial solution is $(Y_0, \theta_0) = (0, 0)$. Non-trivial solutions must satisfy an equation of the form:

$$C_0 s^4 + C_2 s^2 + C_4 = 0$$

Evaluate expressions for C_0, C_2 and C_4. [10%]

(d) The mass distribution along the length of the boat is such that $I = 9S_x^2/8m$. What are the signs (positive or negative) of C_0, C_2 and C_4? Hence deduce the condition on C_0, C_2 and C_4 which leads to instability. [20%]

(e) Calculate an expression for the critical velocity at which instability starts. Comment on the effect of the relative positions of the centre of mass and the aerodynamic centre. [20%]

(f) In a more detailed analysis of this instability, what other factors should be taken into account? [10%]

END OF PAPER

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EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla\phi \text{ and } \nabla^2\phi = 0$$

Bernoulli's equation

for inviscid flow $\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \frac{\partial \phi}{\partial t} = \text{constant throughout flow field.}$

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u,v,w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x,\theta,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma\left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 \eta}{\partial \theta^2}\right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is unstable / stable to inviscid axisymmetric disturbances if Γ^2 decreases / increases with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu\left(\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2}\right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2} = 0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + (1 - \alpha(T - T_0))\boldsymbol{g} + \upsilon \nabla^2 \boldsymbol{u}$$

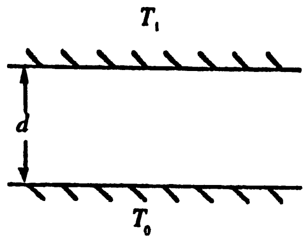
and

$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

Rayleigh-Bénard convection

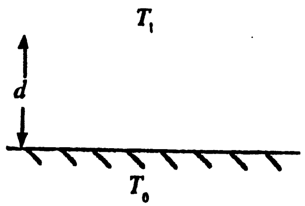
A fluid between two rigid plates is unstable when

$$\text{Ra} \geq 1708$$



A liquid with a free upper surface is unstable when

$$\frac{\text{Ra}}{\text{Ra}_c} + \frac{\text{Ma}}{\text{Ma}_c} \geq 1$$



where

$$\text{Ra} = \frac{g\alpha(T_0 - T_1)d^3}{\upsilon\kappa}, \quad \text{Ma} = \frac{\chi(T_0 - T_1)d}{\rho\upsilon\kappa} \quad \text{with } \chi = -\frac{d\sigma}{dT}$$

$\text{Ra}_c \approx 670 \qquad \text{Ma}_c \approx 80.$

USEFUL MATHEMATICAL FORMULA

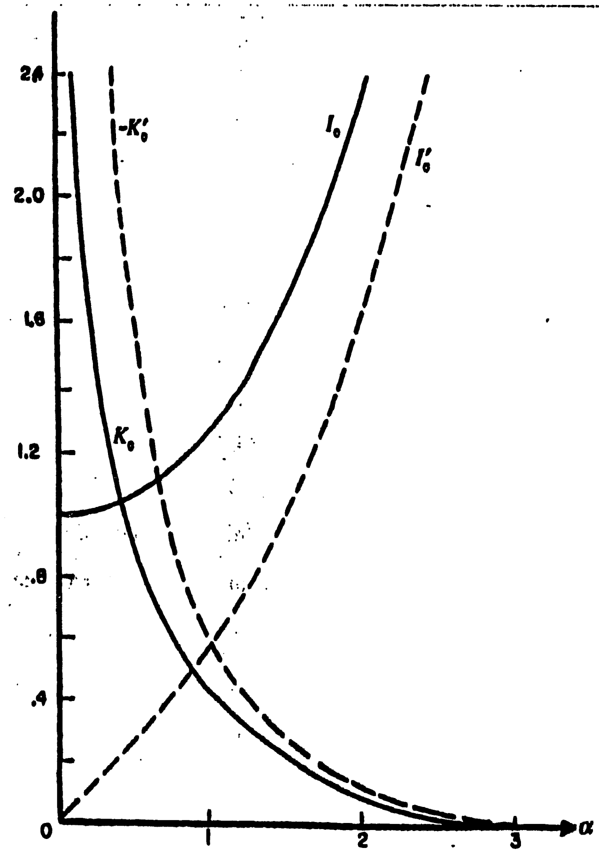
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - k^2 f = 0.$$

$I_0(kr)$ is finite at $r = 0$ and tends to infinity as $r \to \infty$,

$K_0(kr)$ is infinite at $r = 0$ and tends to zero as $r \to \infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative