ENGINEERING TRIPOS PART IIB

Friday 6 May 2005 9 to 10.30

Module 4A12

TURBULENCE

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
Special datasheets (2 pages).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 A model equation for the transport of turbulent kinetic energy in high Reynolds number flow may be written in simplified form as

$$\frac{Dk}{Dt} = K_m\left(\frac{\partial U_i}{\partial x_j}\right)^2 + \text{"diffusion"} - \varepsilon .$$

Identify the individual terms, explaining what they represent.

In a conventional two equation (k-ε) turbulence model the turbulent or eddy viscosity, K_m, is given by $K_m = C_\mu k^2/\varepsilon$. Estimate the value of C_μ in a local equilibrium boundary layer for which the Reynolds stress $-\overline{u'v'} = K_m\, \partial U/\partial y$ where U is the mean velocity and y the wall normal distance. You may assume that $\overline{u'v'}/k = -0.3$. [50%]

For decaying turbulence sufficiently far downstream of a fixed grid, the turbulent kinetic energy k varies approximately as the inverse of x, the distance downstream from the grid. The integral length scale L of the turbulence is proportional to $(x + \text{constant})^{1/2}$. Show that the turbulent kinetic energy per unit wave number, $E(k)$, for an eddy size within the inertial subrange decays faster than the total turbulent kinetic energy. [50%]

2 A two-dimensional jet issues into stationary ambient fluid. The entrainment velocity $u_e = \alpha U$, where U is the local jet velocity, and this is assumed constant over the jet width. Using the similarity assumption that any mean property of the jet varies only as x^β, where x is the streamwise coordinate, determine appropriate values of β to describe:

(i) the jet width; [10%]
(ii) the jet velocity; [20%]
(iii) the largest scale of turbulence; [10%]
(iv) the dissipation of turbulent kinetic energy per unit mass ε; and [20%]
(v) the smallest scale of turbulence (the Kolmogorov microscale). [20%]

You may assume that all jet properties are independent of the width of the jet and that u'/U is constant along the jet. Comment on these assumptions. [20%]

3 (a) Sketch Burgers' vortex showing the irrotational straining motion and the vorticity. Why is Burgers' vortex thought to be important in turbulence? [10%]

(b) Consider the vortex sheet

$$\boldsymbol{\omega} = \omega_o \exp\left[-x^2/\delta^2\right]\hat{\mathbf{e}}_z$$

where ω_o and δ are constants. This corresponds to the velocity field $\mathbf{u}_\omega = u_y(x)\hat{\mathbf{e}}_y$. Show that $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}_\omega$ and $(\mathbf{u}_\omega\cdot\nabla)\boldsymbol{\omega}$ are both zero. [20%]

(c) The vortex sheet sits in the irrotational strain field,

$$\mathbf{u}_s = (-\alpha x, 0, \alpha z)$$

where α is a constant. Show that, if δ is chosen appropriately, this constitutes an exact solution of the steady vorticity equation

$$(\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$

and find an expression for δ in terms of ν and α. [40%]

(d) The strain field is suddenly removed, so that the sheet thickness, δ, becomes a function of time. What is the governing equation for the unsteady vortex sheet? Describe how you would expect δ to vary with time. [30%]

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4 (a) What are Helmholtz's laws of vortex dynamics? [10%]

(b) Starting with the inviscid vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u},$$

show that the helicity density, $\mathbf{u} \cdot \boldsymbol{\omega}$, in an inviscid fluid satisfies

$$\frac{D}{Dt}(\mathbf{u} \cdot \boldsymbol{\omega}) = \nabla \cdot \left[\left(u^2/2 - p/\rho \right) \boldsymbol{\omega} \right].$$

Hence show that the volume integral of $\mathbf{u} \cdot \boldsymbol{\omega}$,

$$H = \int \mathbf{u} \cdot \boldsymbol{\omega} dV,$$

is conserved for a localised distribution of vorticity. [40%]

(c) Consider two, thin, interlinked vortex tubes as show in Figure 1. They have volumes V_1 and V_2, vorticity fluxes Φ_1 and Φ_2, and centre-lines C_1 and C_2. Confirm that the net helicity is given by

$$H = \oint_{C_1} \mathbf{u} \cdot (\Phi_1 d\mathbf{l}) + \oint_{C_2} \mathbf{u} \cdot (\Phi_2 d\mathbf{l})$$

and use Stokes' theorem to find an expression for H in terms of Φ_1 and Φ_2. [30%]

(d) If the direction of the vorticity in one of the tubes was reversed, how would your expression for H change? [10%]

(e) Why would you expect helicity not to be conserved if the fluid were (i) viscous, or (ii) subject to a buoyancy force? [10%]

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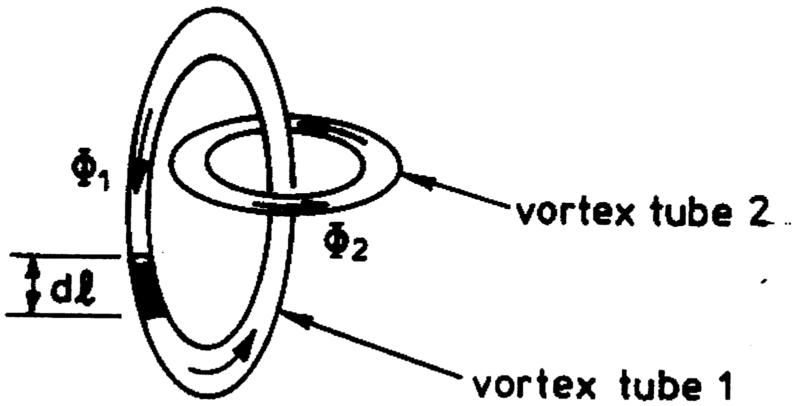

Fig. 1 Two interlinked vortex tubes.

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4A12 Turbulence

Integral length-scale:

$$l = \frac{\int_0^\infty \overline{u'(\mathbf{x},t)u'(\mathbf{x}+\mathbf{r},t)}dr}{\overline{u'(\mathbf{x},t)u'(\mathbf{x},t)}}$$

Reynolds number of turbulence:

$$R_t = \frac{u'l}{\nu}$$

where u' is the rms of velocity fluctuations $u' = \sqrt{\overline{u_i'u_i'}}$.

Reynolds stresses: $-\rho\overline{u_i'u_j'}$

Navier-Stokes equation for the mean flow

$$\rho\frac{\partial U_i}{\partial t} + \rho U_j\frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(-\rho\overline{u_j'u_i'}) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right]$$

Transfer of energy between mean flow and fluctuations: $-\rho\overline{u_i'u_j'}e_{ij}$

Viscous dissipation per unit mass: $\epsilon = 2\,\nu\overline{e_{ij}'e_{ij}'} \sim u'^3/l$

Spectral decomposition of the kinetic energy density;

$$E = \int_0^\infty E(k)dk$$

Kolmogorov cascade in the inertial range:

$$E(k) \sim \epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$$

Kolmogorov micro-scale of dissipation:

$$\eta \sim \frac{\nu^{\frac{3}{4}}}{\epsilon^{\frac{1}{4}}} \sim \frac{l}{R_t^{3/4}}$$

The *log* region in the turbulent boundary layers:

$$\frac{U}{u_*} = A\ln y_+ + B$$

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4A12 Vortex Dynamics

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot dS$

Stokes : $\int (\nabla \times A) \cdot dS = \oint A \cdot dl$

Vector Identities

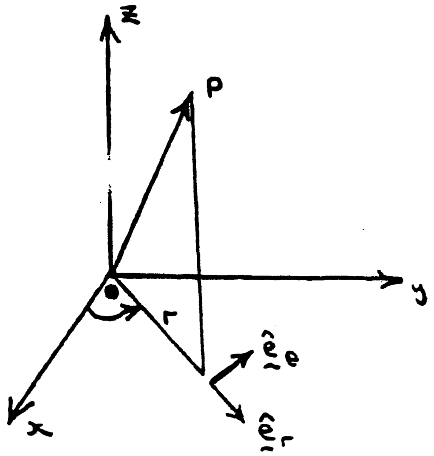
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

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