ENGINEERING TRIPOS PART IIB

Monday 2 May 2005 9 to 10.30

Module 4C7

RANDOM AND NONLINEAR VIBRATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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1 The linear oscillator shown in Fig. 1 has a mass m, supported by a spring of stiffness k and light damping c. The displacement of the oscillator at time t is $y(t)$. The oscillator can be excited by force $f(t)$ or by displacement of its base $x(t)$.

(a) If the displacement input is zero and the force input is white noise, with spectral density:

$$S_{ff}(\omega) = S_0 \qquad -\infty < \omega < \infty,$$

calculate the mean square displacement of the mass $E[y^2]$. Hence show that the *mean square bandwidth* $\Delta\omega$ is given by:

$$\Delta\omega = \pi\zeta\omega_n$$

where ζ is the damping ratio and ω_n is the undamped natural frequency of the oscillator. [30%]

The input force is now set to zero and the base is excited by a displacement $x(t)$ with spectral density

$$S_{xx}(\omega) = \frac{S_1}{1+(\omega/\omega_0)^2} \qquad -\infty < \omega < \infty.$$

Since the damping is light, it may be assumed that the response of the oscillator is essentially in a narrow band of frequencies close to the natural frequency ω_n.

(b) Starting from the standard Rayleigh distribution for the probability density of peak heights a of a narrow band process:

$$p_p(a) = \frac{a}{\sigma_y^2}\exp\left\{-\frac{a^2}{2\sigma_y^2}\right\} \qquad (a \geq 0)$$

show that the probability of a peak exceeding a given level a is:

$$\Pr(y > a) = \exp\left\{-\frac{a^2}{2\sigma_y^2}\right\} \qquad (a \geq 0)$$ [20%]

(c) Use the mean square bandwidth to estimate the mean square velocity $E[\dot{y}^2]$. [30%]

(d) Using the results of (b) and (c), derive an approximate expression for the amount of viscous damping c required in order to achieve a given value p for the probability that a peak in the velocity $\dot{y}(t)$ exceeds a given level v [20%]

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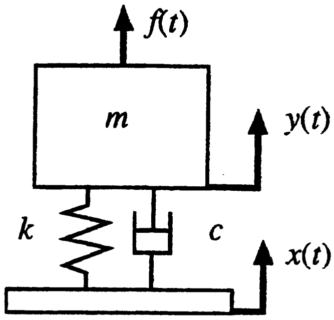

Figure 1

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2 Figure 2 shows a simple single degree-of-freedom model of a vehicle which is moving at constant speed V over a rough road surface, with displacement profile $y(x)$. The vehicle has mass m, and suspension of linear stiffness k and viscous damping c. The displacement of the vehicle body is $z(t)$.

The spatial correlation function $R_{yy}(X)$ and corresponding spectral density $S_{yy}(\gamma)$ of the road profile $y(x)$ are given by:

$$R_{yy}(X) = E[y(x)\, y(x+X)]$$

$$S_{yy}(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yy}(X) e^{-i\gamma X}\, dX$$

where X is a displacement and γ is wavenumber $= 2\pi/\text{wavelength}$.

(a) Show that at angular frequency ω, the spectral density $S_{yy}(\omega)$ of the vertical displacement $y(t)$ of the tyre contact point P is given by

$$S_{yy}(\omega) = \frac{1}{V} S_{yy}(\bar{\gamma})$$

where $\bar{\gamma} = \omega / V$. [30%]

(b) The autocorrelation function of the road profile is measured to be

$$R_{yy}(X) = Ae^{-b|X|} \qquad -\infty < X < \infty$$

where A and b are constants. Show that the road profile spectral density observed by the moving vehicle is

$$S_{yy}(\omega) = \left(\frac{A}{bV\pi}\right)\left(\frac{1}{1+(\omega/bV)^2}\right).$$ [30%]

(c) Calculate the spectral density $S_{ff}(\omega)$ of the dynamic tyre force (i.e. the combined force in the spring and damper). [20%]

(d) Estimate the mean square value of the dynamic tyre force. [20%]

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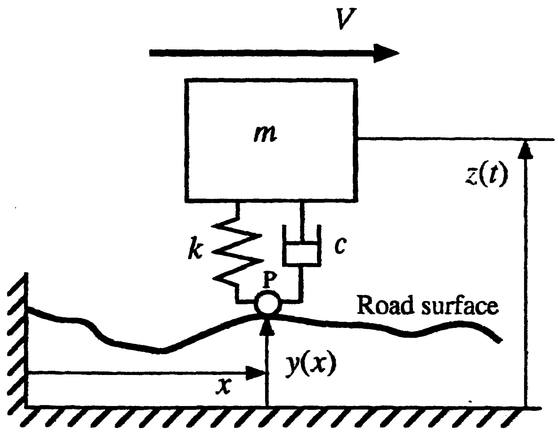

Figure 2

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3 A nonlinear oscillator driven with a sinusoidal force obeys the differential equation

$$m\ddot{x} + k(x) = f\cos\omega t$$

where the nonlinear element $k(x)$ has the force-displacement characteristic shown in Fig. 3. It has a region at small displacement amplitude that describes a 'negative spring' of stiffness $-s$, while for larger amplitudes it shows a positive stiffness s.

(a) For sinusoidal displacement with amplitude X, sketch the force waveform produced by the nonlinear element. [15%]

(b) Determine the Describing Function for the nonlinear element. [40%]

(c) Find the limiting form of the Describing Function for very large amplitude X, and explain this in terms of Fig. 3. [20%]

(d) Find an approximate expression which determines the relation between the response amplitude X and frequency ω for forced motion of the oscillator. (A closed-form expression is not expected.) [25%]

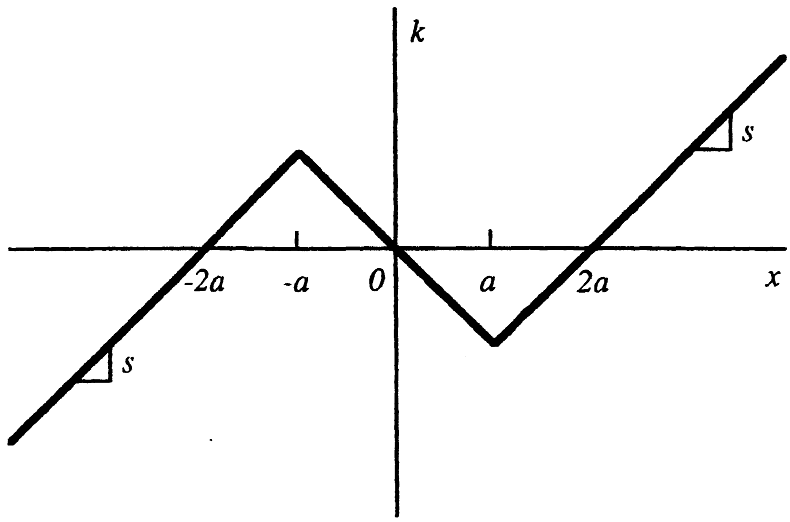

Figure 3

4 The motion of a nonlinear oscillator is described by the differential equation

$$\ddot{x} + kx - x^2 = 0$$

where k is a constant.

(a) Assuming that $k > 0$, find the equilibrium or critical points, and determine their type and stability. Sketch the behaviour of the system in the phase plane. [25%]

Assuming that the system is conservative, find a relation between the potential energy function and the displacement, x. [15%]

(b) Now assume that $k < 0$, so that the system has a negative linear spring. Determine the effect on the critical points and their stability. [20%]

(c) A linear damping term is added to the system from part (b), so that the differential equation becomes

$$\ddot{x} + b\dot{x} + kx - x^2 = 0$$

where b is the damping constant. Find the critical points and investigate their type and stability. Sketch the phase portrait for each distinct case found. [40%]

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