ENGINEERING TRIPOS PART IIB

Wednesday 27 April 2005 2.30 to 4

Module 4C9

CONTINUUM MECHANICS

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

Attachments: Special datasheet (3 pages).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) The stress tensor at a certain place in a body, expressed in a set of orthogonal axes (x_1, x_2, x_3), is given by

$$\sigma_{ij} = \begin{bmatrix} 3 & 4 & -8 \\ 4 & -2 & 6 \\ -8 & 6 & 5 \end{bmatrix} \text{MPa.}$$

(i) What is the value of the stress vector acting on a surface which, at this point, is described by a normal vector which has components, expressed in the same axes, in the ratio $2:1:3$. [25%]

(ii) A second set of axes (x_1', x_2', x_3') is defined in relation to the first by the rotation matrix

$$a_{ij} = \begin{bmatrix} 0.866 & -0.354 & 0.354 \\ 0.5 & 0.612 & -0.612 \\ 0 & 0.707 & 0.707 \end{bmatrix}.$$

Find the numerical value of σ_{23}'. [25%]

(b) A body is subject to a single load T which is steadily increased until the limit load T^L is reached.

(i) State the upper bound theorem for T^L. Discuss carefully the assumptions made. [20%]

(ii) If the body is elastic perfectly-plastic and yields according to the von Mises criterion, using the J_2-flow theory, show that

$$\dot{\varepsilon}_{ij} = \frac{3}{2}\frac{\dot{\varepsilon}_e s_{ij}}{\sigma_y} \quad \text{and} \quad \int_V \sigma_{ij}\dot{\varepsilon}_{ij}\mathrm{d}V = \sigma_y \int_V \dot{\varepsilon}_e \mathrm{d}V$$

where V is the volume of the body, σ_y is the tensile yield stress, s_{ij} is the deviatoric stress tensor and ε_e is the equivalent strain. [30%]

2 Confirm that the function

$$\phi = C\left[x^2 + y^2 - \frac{1}{a}\left(x^3 - 3xy^2\right) - ka^2\right],$$

in which C and k are constants, can be used as a Prandtl stress function to investigate the elastic torsion of a prismatic bar whose cross-section is an equilateral triangle of side $2a/\sqrt{3}$, as illustrated in Fig. 1, provided that the constant k takes a specific numerical value which should be found. The origin of the axes is coincident with the centroid of the cross-section. [40%]

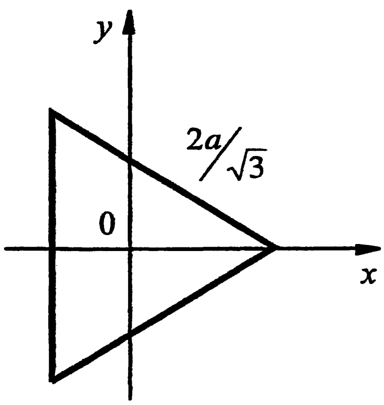

Fig. 1

(a) If $a = 0.15$ m, shear modulus G = 80 GPa and the rate of twist per unit length is to be $5\,^\circ\mathrm{m}^{-1}$, find the value of the shear stress at the mid-point of each side of the section. [30%]

(b) Confirm that the function

$$\Psi = B\left(y^3 - 3x^2y\right),$$

in which B is a constant, can be used to describe the warping of a section of the bar. Find the value of B in terms of the dimension a. [30%]

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3 A thin-walled cylinder is subjected to a combined loading of torsion and longitudinal tension. An infinitely small material element taken from the wall of the cylinder has longitudinal stress σ_{zz}, shear stress $\sigma_{\theta z}$, longitudinal strain ε_{zz} and engineering shear strain $\gamma_{\theta z}(\equiv 2\varepsilon_{\theta z})$ in the cylindrical coordinate system. In uniaxial tension, the material of the cylinder follows a bi-linear stress versus strain relationship with its modulus before and after initial yielding represented by E and F respectively. The material yields according to the von Mises criterion with initial yield stress σ_y.

(a) Describe how the yielding criterion in uniaxial tension, $\sigma = \sigma_y$, is generalised to obtain the von Mises criterion for general three-dimensional problems. Discuss carefully the assumptions used. [20%]

(b) Express explicitly the bi-linear stress strain relationship in uniaxial tension, and hence determine the secant modulus E_s and tangent modulus E_t. [25%]

(c) Using the J_2-flow theory find the incremental constitutive relations governing $d\varepsilon_{zz}$ and $d\gamma_{\theta z}$. [30%]

(d) Loading of the cylinder is proportional, with the ratio $\sigma_{zz}/\sigma_{\theta z}$ fixed at a value of $\sqrt{3}$. When σ_{zz} reaches σ_y, determine the corresponding strain component ε_{zz}. [25%]

END OF PAPER

ENGINEERING TRIPOS Part IIB

Module 4C9 Data Sheet

SUBSCRIPT NOTATION

Repeated suffix implies summation

$\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$	$a_i e_i$
$\underline{a} \bullet \underline{b}$	$a_i b_i \equiv a_i b_j \delta_{ij}$
$\underline{c} = \underline{a} \times \underline{b}$	$c_i = e_{ijk} a_j b_k$
$\underline{d} = \underline{a} \times (\underline{b} \times \underline{c})$	$d_k = -e_{ijk} e_{irs} a_j b_r c_s = a_j b_k c_j - a_i b_i c_k$
Kronecker delta δ_{ij}	$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
e_{ijk}	$e_{ijk} = 1$ when indices cyclic; $= -1$ when indices anticyclic and $= 0$ when any indices repeat
$e - \delta$ identity	$e_{ijk} e_{ilm} \equiv \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$
trace$a$	tr$a = a_{ii} = a_{11} + a_{22} + a_{33}$
$\dfrac{\partial \sigma_{ij}}{\partial x_i} = \dfrac{\partial \sigma_{1j}}{\partial x_1} + \dfrac{\partial \sigma_{2j}}{\partial x_2} + \dfrac{\partial \sigma_{3j}}{\partial x_3}$	$\sigma_{ij,i}$
gradϕ $= \nabla \phi$	$\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}$
div$\underline{V}$	$V_{i,i}$
curl$\underline{V} \equiv \underline{\nabla} \times \underline{V}$	$e_{ijk} V_{k,j}$

Rotation of Orthogonal Axes

If $01'2'3'$ is related to 0123 by rotation matrix a_{ij}

vector v_i becomes	$v'_\alpha = a_{\alpha i} v_i$
tensor σ_{ij} becomes	$\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$

Evaluation of principal stresses

deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$I_1 = \sigma_{ii} = \mathrm{tr}\sigma$

$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$

$I_3 = \frac{1}{6}\left(e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\right)$

$s^3 - I_1' s^2 + I_2' s - I_3' = 0$

$I_1' = s_{ii} = \mathrm{tr}s \; ; \; I_2' = \frac{1}{2}s_{ij}s_{ij} \; ; \; I_3' = \frac{1}{3}s_{ij}s_{jk}s_{ki}$

equilibrium $\sigma_{ij},_i + b_j = 0$

small strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

compatibility $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{lj,ki} - \varepsilon_{ki,lj} e_{pik} e_{qjl} \varepsilon_{ij,kl} = 0$

equivalent to $e_{pik}e_{qjl}\varepsilon_{ij,kl} \equiv e_{pik}e_{qjl}\dfrac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = 0$

Linear elasticity $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Hooke's law $E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}$

Lamé's equations $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Elastic torsion of prismatic bars

Warping function $\Psi(x_1, x_2)$ satisfies $\nabla^2\Psi = \Psi_{,ij} = 0$

If Prandtl stress function $\phi(x_1, x_2)$ satisfies $\nabla^2\phi = \phi_{,ij} = -2G\alpha$ where α is the twist per unit length then

$$\sigma_{31} = \phi,_2 = \frac{\partial\phi}{\partial x_2} \;,\; \sigma_{32} = -\phi,_1 = -\frac{\partial\phi}{\partial x_1} \text{ and } T = 2\iint_A \phi(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Equivalence of elastic constants

	E	ν	$G=\mu$	λ
E, ν	–	–	$\frac{E}{2(1+\nu)}$	$\frac{\nu E}{(1+\nu)(1-2\nu)}$
E, G	–	$\frac{E-2G}{2G}$	–	$\frac{(2G-E)G}{E-3G}$
E, λ	–	$\frac{E-\lambda+R}{4\lambda}$	$\frac{E-3\lambda+R}{4}$	–
ν, G	$2G(1+\nu)$	–	–	$\frac{2G\nu}{1-2\nu}$
ν, λ	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\frac{\lambda(1-2\nu)}{2\nu}$	–
G, λ	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda}{2(\lambda+G)}$		–

$R=\sqrt{E^2+2E\lambda+9\lambda^2}$

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