ENGINEERING TRIPOS PART IIB
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Friday 6 May 2005 14.30 to 16.00

Module 4C14

MECHANICS OF BIOLOGICAL SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
Special datasheet (2 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) Summarise the main mechanical functions of the cytoskeleton within a cell, by reference to the microtubules, actin cortex and intermediate filaments. [30%]

(b) Outline the structure of a *sarcomere* and explain the role of the thick filaments, thin filaments and the Z-discs. [30%]

(c) What is the significance of the *persistence length* in dictating the properties of biological fibres? [20%]

(d) Explain the significance of the stiffness of the wall of a blood vessel in determining the blood flow rate. [20%]

2 A two dimensional biological network has the microstructure shown in Fig. 1. It can be treated as a periodic assembly of elastic-ideally plastic struts, with cell wall Young's modulus E_S and yield strength σ_{YS}. The struts are of uniform thickness t.

(a) Obtain an expression for the relative density $\bar{\rho}$ in terms of t and the cell size ℓ. [15%]

(b) Calculate the effective modulus E_2 and the effective yield strength σ_{2Y} along the x_2-direction. The contribution from bending of the struts may be neglected. [30%]

(c) The effective strength of the network along the x_1-direction is dictated by plastic bending of the hinges at the ends of each inclined strut.

(i) Use a velocity diagram to relate the rate of rotation of the inclined struts to the resulting macroscopic strain rate $\dot{\varepsilon}_{11}$ along the x_1-direction. [15%]

(ii) Use virtual work to obtain an expression for the macroscopic effective strength σ_{1Y} along the x_1-direction in terms of the relative density and the cell wall strength σ_{YS}. [20%]

(iii) Calculate the nominal tensile strain along the x_1-direction in order for the structure to switch from a bending dominated response to a stretching dominated response. [20%]

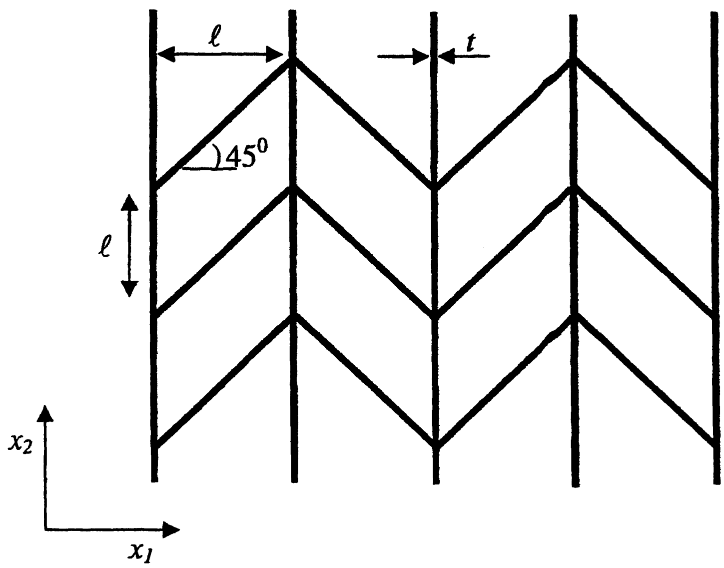

Fig. 1

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3 (a) Physiologists have long known that muscle speed v decreases with increasing load T according to the Hill equation

$$(T+a)v = b(T_o - T)$$

where a and b are constants and T_o is the isometric tetanic tension.

(i) Use the Hill equation to derive an expression for the power output of the muscle as a function of the muscle load T. [10%]

(ii) Determine the muscle speed v at which the muscle power is maximised. What is the implication of this in selecting a gear on a bicycle when riding up an incline? [30%]

(b) In the Huxley sliding filament model for a muscle, the fraction $n(x)$ of attached crossbridges is given by

$$n(x) = \begin{cases} n_o \exp(kx/v) & x < 0 \\ n_o & 0 \le x \le h \\ 0 & x > h \end{cases}$$

where n_o and k are constants, x is the position of an actin binding site from the equilibrium position of a myosin head and $v \equiv -\mathrm{d}x/\mathrm{d}t$ is the shortening velocity of the muscle. The muscle under consideration has a cross-sectional area A, sarcomere length s, and m crossbridges per unit volume. Assume that a linear spring with stiffness λ connects the myosin head to the thick filament.

(i) Determine the tension-velocity relation for this muscle. You may assume that the myosin sites M and the actin sites A have a separation l which is much greater than h. [40%]

(ii) Sketch the tension-velocity relation determined above and briefly discuss the quality of the agreement of this model with the Hill equation. [20%]

Hint: $\int x e^{qx} \mathrm{d}x = \frac{1}{q^2}\left[qxe^{qx} - e^{qx}\right]$

4 (a) Describe the role of the respiratory system and its principle of operation. [20%]

(b) Describe and account for the variations of partial pressure of O_2 and CO_2 in the blood when entering and leaving the alveolar capillaries. [20%]

(c) Explain the mechanism of CO_2 removal from the blood. [20%]

(d) Oxygen is dissolved at concentration U uniformly across the cross-section of a capillary; steady state conditions are assumed such that U is independent of time t but can vary with the axial co-ordinate x along the capillary, of length L, uniform cross-sectional area A and perimeter C. The partial pressure P of the gas and the velocity v of blood flow can be taken as constant along the capillary. Diffusion of oxygen across the capillary wall is governed by

$$\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial x} = D(\sigma P - U)$$

where D is the diffusion constant and σ is the solubility of oxygen in blood.

(i) Obtain an expression for $U(x)$, with the end condition $U(0) = U_0$. [20%]

(ii) Use mass conservation to calculate the rate of loss of oxygen across the wall of an infinitely long capillary. [20%]

END OF PAPER

Paper G4: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \frac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \frac{1}{2r}\frac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2 - \frac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \frac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2 + \frac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \frac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \frac{\partial u}{\partial x}$	$\varepsilon_{rr} = \frac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \frac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$	$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}$
Compatibility	$\frac{\partial^2\gamma_{xy}}{\partial x\partial y} = \frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\frac{\partial}{\partial r}\left\{r\frac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \frac{\partial}{\partial r}\left\{r^2\frac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\frac{\partial\varepsilon_{rr}}{\partial r} + \frac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity)	$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0$	$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x} = 0$	$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\} \times \left\{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{dF}{dy}$, $\sigma_{zy}\ (= \tau_y) = -\dfrac{dF}{dx}$

Equilibrium: $T = 2\int_A \mathrm{F}dA$

Governing equation for elastic torsion: $\nabla^2\mathrm{F} = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^{\mathrm{T}}\,[D]\,\underset{\sim}{\varepsilon}\,\mathrm{d}V$, $\mathrm{W} = \underset{\sim}{P}^{\mathrm{T}}\underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - \mathrm{I}_1\,\sigma_P^2 + \mathrm{I}_2\sigma_P - \mathrm{I}_3 = 0$ where $\mathrm{I}_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$\mathrm{I}_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad \mathrm{I}_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\,\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\}^{1/2}$

Equivalent strain increment $\mathrm{d}\bar{\varepsilon} = \sqrt{\frac{2}{3}}\,\{\mathrm{d}\varepsilon_1^2 + \mathrm{d}\varepsilon_2^2 + \mathrm{d}\varepsilon_3^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $\mathrm{d}\varepsilon_1 : \mathrm{d}\varepsilon_2 : \mathrm{d}\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{\mathrm{d}\varepsilon_1}{\sigma'_1} = \frac{\mathrm{d}\varepsilon_2}{\sigma'_2} = \frac{\mathrm{d}\varepsilon_3}{\sigma'_3} = \frac{\mathrm{d}\varepsilon_1-\mathrm{d}\varepsilon_2}{\sigma_1-\sigma_2} = \frac{\mathrm{d}\varepsilon_2-\mathrm{d}\varepsilon_3}{\sigma_2-\sigma_3} = \frac{\mathrm{d}\varepsilon_3-\mathrm{d}\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{\mathrm{d}\bar{\varepsilon}}{\bar{\sigma}}\ .$$