ENGINEERING TRIPOS   PART IIB

Wednesday 27 April 2005   2.30 to 4

Module 4F1

CONTROL SYSTEM DESIGN

Answer not more than two questions.

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Formulae sheet (3 pages).

Supplementary pages:
Two extra copies of Fig. 1 (Question 3).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) For a rational transfer-function $G(s)$, a close relationship can be described between the root-locus diagram and the Nyquist diagram.

(i) Write down the mapping between two complex planes which establishes this relationship, and describe how the two diagrams are defined in terms of this mapping. For the root-locus diagram, justify that this characterisation agrees with the definition of the root-locus. [20%]

(ii) State and prove the rule which determines which segments of the real axis are part of the root-locus diagram. [20%]

(iii) State the definition of a conformal mapping. State an important property of conformal mappings, and explain the connection with breakaway points in the root-locus diagram. [20%]

(b) Consider the plant

$$G(s) = \frac{1}{(s+1)^2(s+2)}.$$

Sketch the Nyquist diagram and the root-locus diagrams for $k > 0$ and $k < 0$ for $G(s)$. Calculate all axis crossings which are relevant in assessing closed-loop stability. Calculate the asymptote centre and breakaway points in the root-locus diagrams. [40%]

2 (a) State the small gain theorem. An uncertain system is modelled as:

$$G_1(s) = (1 + \Delta(s))G(s)$$

where $G(s)$ is a known transfer function and $\Delta(s)$ is assumed only to be stable and to satisfy a bound $|\Delta(j\omega)| \leq h(\omega)$ for all ω. Let $K(s)$ stabilise $G(s)$ in a unity negative feedback system. Derive a necessary and sufficient condition for $K(s)$ to stabilise $G_1(s)$. [20%]

(b) Consider a plant with transfer function

$$\frac{s + 1.25}{s^2 - 1} + \frac{(\delta_1 s + \delta_2)(s + 2)}{s^2 - 1}$$

where the parameters δ_1 and δ_2 are unknown.

(i) Find a constant controller $K(s) = k$ which stabilises the nominal plant with $\delta_1 = \delta_2 = 0$ and achieves critical damping (coincident poles) with fastest possible decay rate for the closed-loop. [20%]

(ii) Express the uncertain plant in the form given in Part (a) and derive the corresponding condition for robust stability when the controller of Part (i) is used. [15%]

(iii) Use the condition of Part (ii) to derive upper bounds on $|\delta_1|$ and $|\delta_2|$ which are sufficient to guarantee robust stability. [25%]

(c) The sensitivity function $S(s)$ of a linear, single-loop control system frequently satisfies an integral relationship which implies that $|S(j\omega)| > 1$ at some frequency ω. Give conditions on a plant where this is true for any stabilising controller which can be designed, and give an example of a plant where it is not true. [20%]

3 Figure 1 is the Bode diagram of a system $G(s)$ for which a feedback compensator $K(s)$ is to be designed. It may be assumed that $G(s)$ is a real-rational transfer function, and that all poles and zeros have moduli which lie within the range of frequencies shown on the diagram.

(a) (i) Sketch on a copy of Fig. 1 the expected phase of $G(j\omega)$ if $G(s)$ had no poles or zeros with $\text{Re}(s) > 0$. [20%]

(ii) Determine whether $G(s)$ has any right half plane poles or any right half plane zeros (it doesn't have both) and estimate their location (if there are any). [20%]

(iii) Comment on any limitations on the achievable crossover frequency that might be faced in a control systems design for this plant. [10%]

(b) If a constant controller $K(s) = k$ is used, determine the number of right half plane poles of the closed-loop system for all values of k, both positive and negative. Justify your answer using a sketch of the complete Nyquist diagram and application of the Nyquist stability criterion. [20%]

(c) Find a compensator $K(s)$ to achieve the following specifications:

A: internal stability of the closed-loop;

B: a phase margin of at least 30^o;

C: reduction of the effects of sensor noise on the plant output by a factor of at least 10 at frequencies above 200 rad/s.

Show, on a copy of Fig. 1, the effect of this compensator on the open-loop transfer function. [30%]

Two copies of Fig. 1 are provided on separate sheets. These should be handed in with your answers.

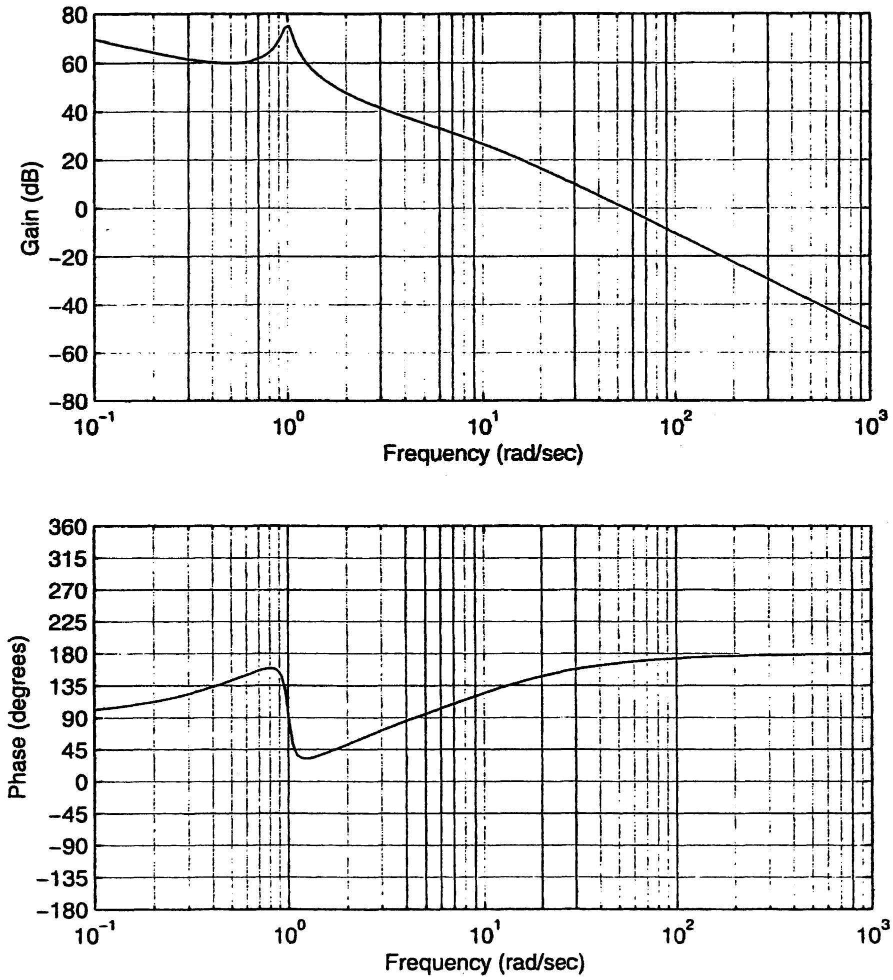


Figure 1: Bode diagram of $G(s)$ for Question 3.

END OF PAPER

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

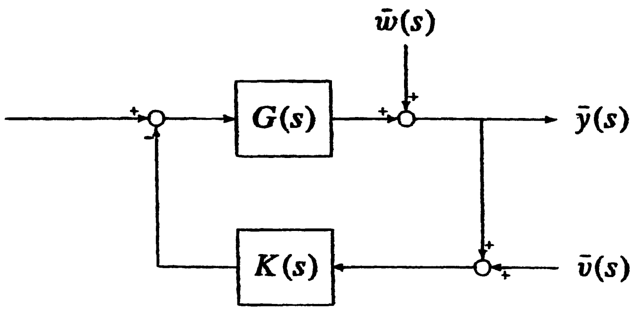
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).

A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90^\circ.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP (Re(s) > 0) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^{v})| \log\coth\frac{|v|}{2}\, dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^{v}.$$

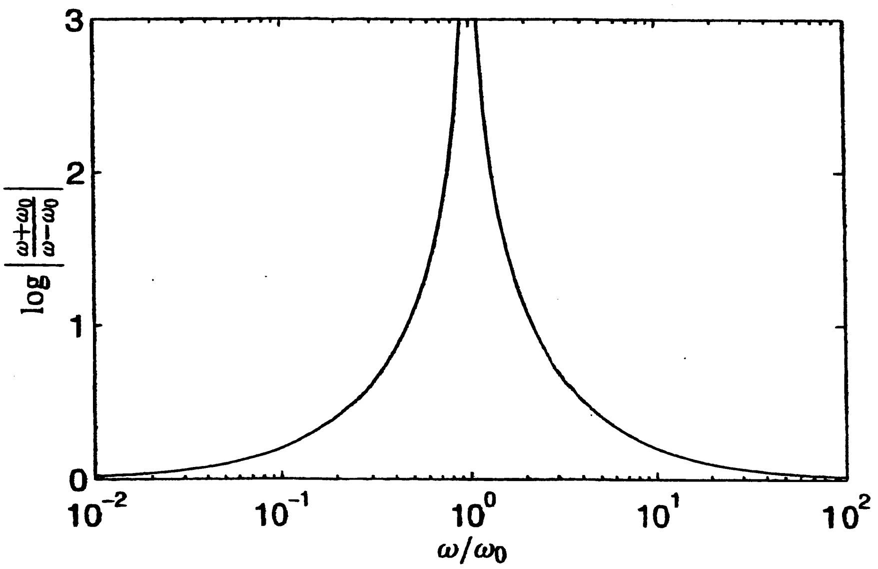


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2} \left.\frac{d\log|L(j\omega_0 e^{v}|)}{dv}\right|_{\omega=\omega_0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\text{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\left(j|s_0|e^{v}\right)\right| dv}$$

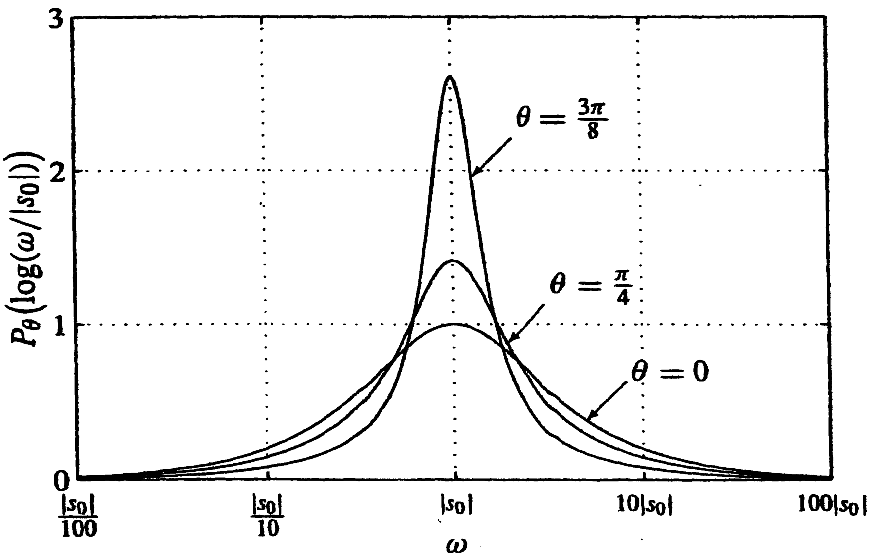
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

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November 2002

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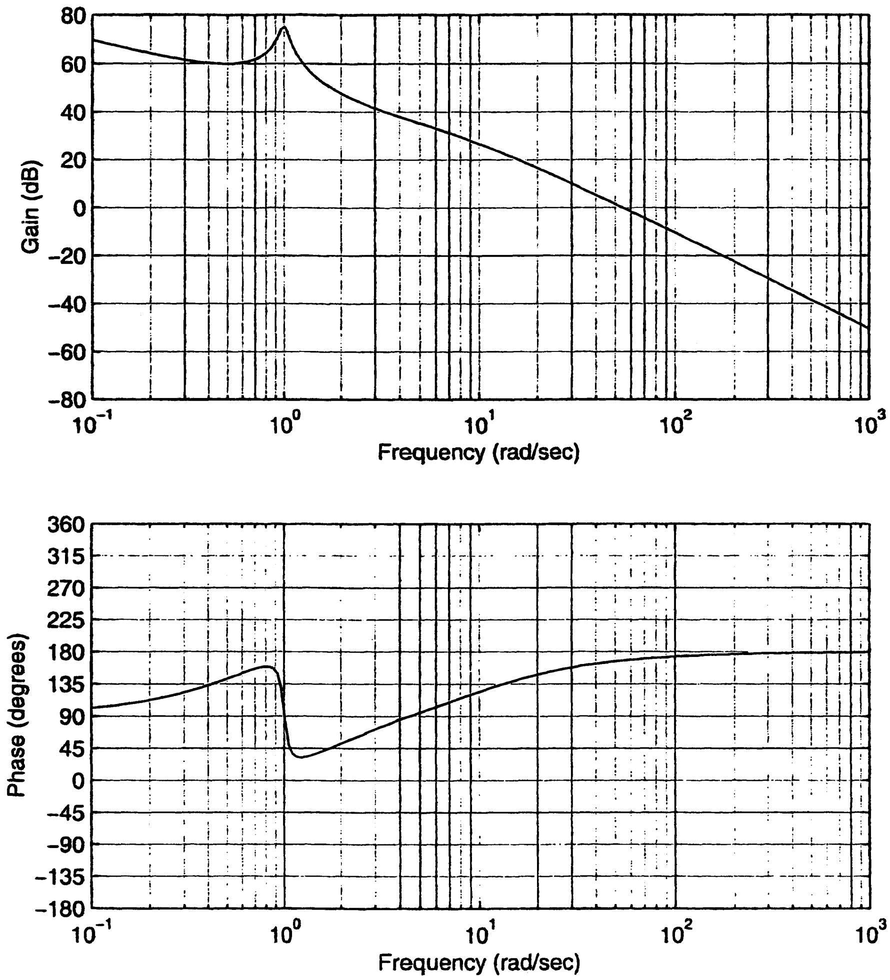

Extra Copy of Fig. 1: Bode diagram of $G(s)$ for Question 3.

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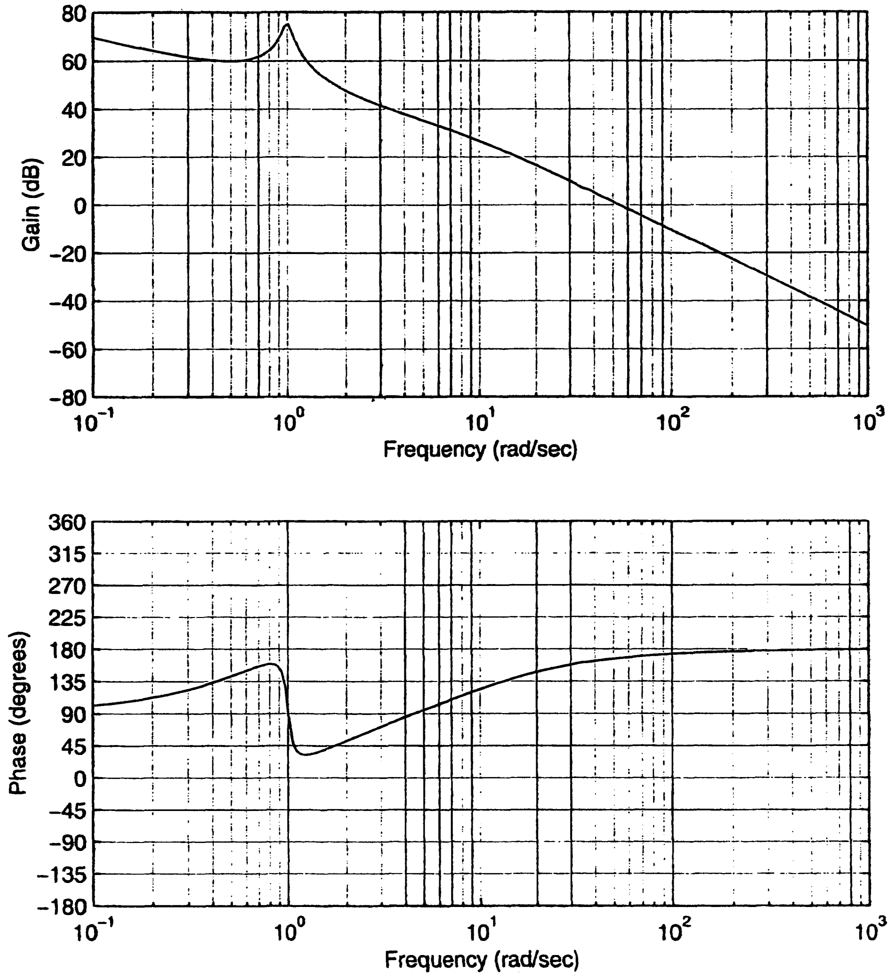
27 April 2005, Module 4F1, Question 3



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