UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Monday 9 May 2005 2.30 to 4

Module 4F3

NONLINEAR AND PREDICTIVE CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) Define the following terms, in the context of a dynamical system

$$\dot{x} = f(x)$$

(i) *Invariant* set [10%]

(ii) *Equilibrium* point [10%]

(iii) *Stable* equilibrium point [10%]

(iv) *Globally asymptotically stable* equilibrium point. [10%]

(b) The following equations arise in an adaptive control problem:

$$\begin{aligned}
\dot{x}_1 &= ax_1 - x_1x_3 + 2u \\
\dot{x}_2 &= -2x_2 + 2u \\
\dot{x}_3 &= bx_1(x_1 - x_2)
\end{aligned}$$

If u is a non-zero constant, find the equilibrium point. [25%]

(c) By considering the function

$$V(x_1, x_2, x_3) = b(x_1 - x_2)^2 + b(x_2 - u)^2 + (x_3 - 2 - a)^2$$

show that the equilibrium point in part (b) is globally asymptotically stable if $b > 0$.

[35%]

2 (a) Figure 1 shows a negative feedback connection of a linear system with transfer function $G(s)$ and a nonlinear gain $\psi(y)$, where y is the output of the linear system. State the *circle criterion* for global asymptotic stability of this feedback system, if $\alpha \leq \psi(y) \leq \beta$. [10%]

(b) It is known that if $G(s)$ is strictly positive real, and $\psi(y) \geq 0$, then the system shown in Fig.1 is globally asymptotically stable. By considering the transformation

$$\psi = \frac{\tilde{\psi} - \alpha}{\beta - \tilde{\psi}} = -\frac{\frac{\alpha - \tilde{\psi}}{\beta - \alpha}}{1 + \frac{\alpha - \tilde{\psi}}{\beta - \alpha}}$$

(or otherwise), derive the circle criterion. (You may assume that the transformation $\tilde{z} = (z - 1)/(\beta - \alpha z)$ maps the imaginary axis to a circle whose centre is real.) [40%]

(c) If $G(s) = 2/(s+1)^2$, $\alpha = 0$ and $\beta = 1$, show that the system shown in Fig.1 is globally asymptotically stable. [30%]

(d) Comment on the relative advantages and disadvantages of the circle criterion and of the describing function method for analysing nonlinear feedback systems. [20%]

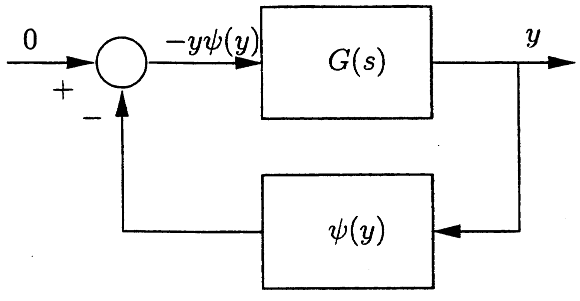

Fig. 1

3 The temperature $x(k)$ (in $^\circ$ C) of a heated pool is described by the equation

$$x(k+1) = 0.2x(k) + 0.5u(k) + 0.8d(k)$$

where k denotes the sample instant (the sampling period is 1 hour), $u(k)$ is the power supplied to the heating element (in kW), and $d(k)$ is the ambient air temperature (in $^\circ$ C). Measurements of the pool and ambient temperatures are available at each sample instant. Assume that the ambient temperature varies so slowly that it can be considered to be constant on a given day.

(a) An unconstrained receding horizon control (RHC) law is given by $u = K(x-r)$ where r is the desired pool temperature, and $K = -0.29$.

Show that the steady-state pool temperature is approximately 19° C if the set-point is kept constant at $r = 26^\circ$ C and the ambient temperature is $d = 18^\circ$ C. [10%]

(b) Suppose that the RHC law is modified to $u = u_\infty + K(x-r)$ where u_∞ is a constant. Find the value of u_∞, as a linear function of r and d, such that the pool reaches the desired temperature, without steady-state error, on any given day. [25%]

(c) Suppose that the power is constrained to lie in the range $0 \le u \le 10$ (kW), and that the pool temperature is constrained to lie in the range $20 \le x \le 30$ ($^\circ$ C). For what range of ambient temperatures does there exist an admissible steady-state input such that the steady-state pool temperature lies between these limits? [30%]

(d) The constraints in part (c) are passed to an optimisation algorithm in the form:

$$J \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \le c + Wx(k) + Yd(k).$$

If a prediction horizon of length 2 is used, find the matrix J and the vectors c, W and Y, assuming that the ambient temperature remains constant over the prediction horizon (ie $d_s = d(k)$ for $s = 0, 1$). [35%]

4 (a) List some of the advantages and disadvantages of predictive control, referring to an industrial application in your answer. [30%]

(b) Consider the following open-loop stable, discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

and the one-step cost

$$V(x, u_0) := x_0^T Q x_0 + u_0^T R u_0 + x_1^T P x_1$$

where $x_0 = x$ is the current, measured value of the state, and the predicted state is given by $x_1 = Ax_0 + Bu_0$. P, Q and R are positive-definite matrices, with the terminal weight P satisfying the Lyapunov equation $P = A^T PA + Q$.

For a given x, let $u_0^*(x)$ denote the input that minimises $V(x, u_0)$, and let $V^*(x) := V(x, u_0^*(x))$ be the minimum value.

(i) By considering $V(Ax + Bu_0^*(x), 0)$, show that [50%]

$$V^*(Ax + Bu_0^*(x)) < V^*(x) \quad \text{for all} \quad x \neq 0$$

Hint: The substitutions $w_0 = Ax + Bu_0^*(x)$ and $w_1 = Aw_0$ simplify the algebra considerably.

(ii) What additional conditions on $V^*(x)$ are needed in order to be able to claim that $V^*(x)$ is a Lyapunov function for the closed-loop system [20%]

$$x(k+1) = Ax(k) + Bu_0^*(x(k))\,?$$

END OF PAPER